STUDY OF GENERATING FUNCTIONS AND SPECIAL FUNCTIONS

ABSTRACT

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy

IN

APPLIED MATHEMATICS

By

AHMED ALI LASMA ATASH

Under the Supervision of

Dr. Abdul Hakim Khan

DEPARTMENT OF APPLIED MATHEMATICS
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ALIGARH, (INDIA)

2005

ABSTRACT

Special functions in general and hypergeometric functions and polynomials in one or more variables in particular occur frequently in a wide variety of problems in physics, engineering, statistics and operations research. In theory of special functions, generation functions, summations and transformations formula have received some attention of some authors during the last years.

Generating functions, finite sum properties and transformations play a crucial role in the study of special functions.

In view of growing importance of generating functions, this thesis contains certain classes of generating functions which are linear, bilinear, bilateral, double and multiple for certain special functions and polynomials in one, two and several variables. Such generating functions are obtained by using series rearrangement method, integral operator techniques, Nishimoto's fractional calculus and group theoretic method.

Some transformations, fractional derivative formulas, finite sum properties, for certain hypergeometric functions and polynomials are also presented and various special cases are deduced.

In our work many known results of Chatterjea [15], Exton [25], Feldheim [27], Kar and Basu [32], Khan and Shukla [36], Majumdar [45], Maxinar [48], Pathan and Bin Saad [67], Pathan and Khan [64], Srivastava [82, 84, 86] and Srivastava and Manocha [94] are shown as special cases of our main results and many new results are also presented.

The present thesis comprises six chapters. A brief summary of the problems is presented at the beginning of each chapter and then each chapter is divided into a number of sections. Because of the close association of special functions, with generating functions, a brief review of these important topics is presented in the ***first chapter***. It provides a systematic introduction to the most of the important special functions that commonly arise in practice and explores many of their salient properties.

This chapter is also intended to make the thesis as much self contained as possible.

Chapter **(2)** deals with the study of Laguerre polynomials of two, three, four and m-variables. Certain new finite sum properties, transformations and generating functions for these polynomials have been obtained in this chapter. A known results given by Exton [25], Srivastava and Manocha [94] and many generating functions involving generalized hypergeometric function ${}_pF_q$, confluent hypergeometric functions of several variale $\Psi_2^{(s)}$, Kampé de Fériet function of two variable $F\begin{matrix} A:B;\, B' \\ C:D;\, D' \end{matrix}[x, y]$ and generalized Kampé de Fériet of several variables $F^{A:B';\cdots\cdots;B^{(n)}}_{C:D';\cdots\cdots;D^{(n)}}[x_1,\cdots\cdots,x_n]$ are shown here to be special cases of our main results.

Chapter **(3)** devoted to obtain double generating functions for Gauss hypergeometric function ${}_2F_1$, generalized hypergeometric function ${}_pF_q$ and generalized Kampé de Fériet function of several variables $F^{A:B';\cdots\cdots;B^{(n)}}_{C:D';\cdots\cdots;D^{(n)}}[x_1,\cdots\cdots,x_n]$.

A further double generating relations for Legendre, Laguerre, Jacobi and Rice polynomials are also obtained.

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In ***Chapter*** **(4)**, we derive some fractional derivatives formulas involving hypergeometric functions of three variables by application of Nishimoto's fractional calculus (N-Fractional Calculus). Certain generating functions (linear, bilinear, bilateral, double and multiple) for triple hypergeometric function $F^{(3)}[x,y,z]$and Saran's functions F_G, F_N, F_S are obtained. Many results (known and new) involving Gauss ${}_2F_1$ and Appell's double hypergeometric functions F_1, F_2, F_3 are obtained as a special cases in this chapter.

In ***chapter*** **(5)**, a new generating functions for triple hypergeometric function $F^{(3)}[x,y,z]$, which are linear and double generating functions is derived by using integral operators. Many special cases involving triple hypergeometric function $F^{(3)}[x,y,z]$, Kampé de Fériet function of two variables $F^{A:B;\,B'}_{C:D;\,D'}[x,y]$ and Appell's double hypergeometric function F_2 are obtained. It is also shown how the main results (5.2.4) and (5.4.3) related to a known results Srivastava [82] and Pathan and Khan[68].

Finally, in ***chapter*** **(6)**, we prove a general theorems on a general class of generating relations involving Legendre, Laguerre and Gegenbauer polynomials with the help of group theoretic method. Importance of these theorems lie in the fact that all particular cases of generating functions can be easily deduced by attributing suitable value to a_n and then making use of known generating functions involving Legendre, Laguerre and Gegenbauer polynomials. Results given by Majumdar [45] and Kar and Basu [32] are shown here to be special cases of our main results.

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CERTIFICATE

This is to certify that the contents of this thesis entitled ***"Study of generating functions and special functions"*** is the original research work of **Mr. Ahmed Ali Lasma Atash** carried out under my supervision. He has fulfilled the prescribed conditions given in the ordinances and regulations of Aligarh Muslim University, Aligarh.

I further certify that the work included in the thesis has not been submitted either partly or fully to any other university or institution for the award of any degree.

Dated: 25-10-05

Dr. Abdul Hakim Khan
(Supervisor)
Department of Applied Mathematics
Z.H. College of Engg. & Technology
Aligarh Muslim University

25/10/2005

Dedicated

to

My Parents

Acknowledgement

All praises and thanks are due to almighty Allah whose help guides me throughout the different stages of this research.

I would like to express my sincere gratitude to my esteemed supervisor. **Dr. Abdul Hakim Khan**, Department of Applied Mathematics, Z.H. College of Engg. & Technology, Aligarh Muslim University, Aligarh, for the inspiration and invaluable guidance for my research work.

I shall always remain grateful to **Prof. Mumtaz Ahmad Khan**, Chairman, Department of Applied Mathematics, Z.H. College of Engg. & Technology, Aligarh Muslim University, Aligarh, for his constant help. His encouragement and inspiration go a long way to improve my work. He left no stone unturned to support me with his invaluable suggestions since the very beginning of my research.

My sincere gratitude is extended to **Prof. M. Mukhtar Ali**, previous Chairman for providing me the necessary research facilities during the period of my research. My sincere gratitude is extended to **Mr. Sirajul Hasan, Mr. Parvez Ali, Mr. Syed Ajaz Kirmani** and **Mr. Madhukar Sharma**, Department of Applied Mathematics for their kindness and cooperation during the period of my research.

I am also grateful to my dear friends ***Mr. Khalid Salah Ajrah*** *from Hyderabad University and* ***Mr. Mohammad Salem Masroor*** *from Delhi University for their cooperation. They supported me with the needed books and materials during my research work.*

I am unable to find words to express my deep and faithful gratitude to my parents in Yemen. Their prayer was the light during the difficult moments, special thanks are due to my brothers, sisters and all my family members who are the source of my inspiration all through.

My sincerest appreciation and remarks of admiration are extended to my wife and to my sons – ***Shaker, Arif, Abdulmajed, Abdullah & Abdulaziez*** *– and my daughter* ***Hanan*** *their great support, care, understanding and prayer were very much needed during the various stages of my research.*

Ah

Dated: 22.10.2005 ***(Ahmed Ali Lasma Atash)***

PREFACE

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Finally, in ***chapter*** **(6)**, we prove a general theorems on a general class of generating relations involving Legendre, Laguerre and Gegenbauer polynomials with the help of group theoretic method. Importance of these theorems lie in the fact that all particular cases of generating functions can be easily deduced by attributing suitable value to a_n and then making use of known generating functions involving Legendre, Laguerre and Gegenbauer polynomials. Results given by Majumdar [45] and Kar and Basu [32] are shown here to be special cases of our main results.

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CONTENTS

CHAPTER-I

Introduction, Definitions and Notations

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INTRODUCTION, DEFINITIONS AND NOTATIONS

1.1 INTRODUCTION

The theory of differential equations has originated a large number of new functions. We call them as special functions.

Theory of special functions plays a basic role in the formalism of mathematical physics and applied mathematics. It covers extremely wide domain of study, firstly formulated by the pioneering works of Euler, Gauss, Laplace, Bessel, Legendre, Jacobi, Hermit, Laguerre and many others, secondly by Whittaker, Watson, Ramanujan, Appell, Ragab, Hardy, Lebdev,Erdelyi, Chaundy, Bailey et cetera and continuously refined by new achievements and suggestion within the context of applied sciences.

This chapter aims at introduction of several classes of special functions which occur rather more frequently in the study of various generating functions of various special functions. We present some basic definitions and important properties of special functions needed for the presentation of the subsequent chapters. In section (1.2) we first give the definition of elementary functions as Gamma function and then proceed to the Gaussian hypergeometric function and their generalization.

While a brief account of other hypergeometric functions of two and several variables are presented in section (1.3) and (1.4) respectively. In section (1.5), we present the definitions and properties of the classical orthogonal polynomials such as Jacobi, Gegenbauer, Legendre and Laguerre polynomials.

A concept of generating functions and their classification is given in the last section (1.6).

1.2 GAUSSIAN HYPERGEOMETRIC FUNCTION

With a view to introducing the Gaussian hypergeometric series and its generalizations, we recall some definitions and identities involving pochhammer's symbol $(\lambda)_n$, Gamma function $\Gamma(z)$ and the related function.

The Gamma Function

One of the simplest but very important special functions is the Gamma function $\Gamma(z)$. It has several equivalent definitions, most of which are due to Euler.

We follow Euler in defining the function $\Gamma(z)$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \ \mathrm{Re}(z)>0 \qquad \text{...... (1.2.1)}$$

Upon integration by parts, definition (1.2.1) yields the recurrence relation

$$\Gamma(z+1) = z\,\Gamma(z) \qquad \text{...... (1.2.2)}$$

From the relation (1.2.1) and (1.2.2) it follows that

$$\Gamma(z) = \begin{cases} \int_0^\infty t^{z-1} e^{-t}\, dt, & \mathrm{Re}(z)>0, \\ \dfrac{\Gamma(z+1)}{z}, & \mathrm{Re}(z)<0, \quad z \neq -1,-2,-3,\cdots . \end{cases} \qquad \text{...... (1.2.3)}$$

The recurrence relation (1.2.2) yields the useful result

$$\Gamma(z+1) = z! \ , \ z = 0,1,2, \ldots\ldots \qquad \ldots\ldots (1.2.4)$$

which shown that the Gamma function $\Gamma(z)$ is a generalization of the function z!.

$$z! = \int_0^\infty t^z e^{-t} \, dt, \ z = 0,1,2, \cdots\cdots \qquad \ldots\ldots (1.2.5)$$

The Pochhammer's Symbol and the Factorial Function

The Pochhammer's symbol $(\lambda)_n$ is defined by

$$(\lambda)_n = \begin{cases} 1, & if \ \ n = 0 \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+n-1) & , if \ \ n = 1,2,3,\ldots\ldots \ . \end{cases} \qquad \ldots\ldots (1.2.6)$$

Since $(1)_n = n!$, $(\lambda)_n$ may be looked upon as a generalization of the elementary factorial, hence the symbol $(\lambda)_n$ is also referred to as the factorial function.

In terms of Gamma function, we have

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \ , \ \lambda \neq 0, -1, -2, \cdots\cdots \qquad \ldots\ldots (1.2.7)$$

Furthermore, the binomial coefficient may be now expressed as

$$\binom{\lambda}{n} = \frac{\lambda \ (\lambda-1) \cdots\cdots (\lambda-n+1)}{n!} = \frac{(-1)^n \ (-\lambda)_n}{n!} \qquad \ldots\ldots (1.2.8)$$

or, equivalently, as

$$\binom{\lambda}{n} = \frac{\Gamma(\lambda+1)}{n!\Gamma(\lambda-n+1)} . \qquad \ldots\ldots (1.2.9)$$

It follows from (1.2.8) and (1.2.9) that $\dfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-n+1)} = (-1)^n \ (-\lambda)_n$,

$$\ldots\ldots (1.2.10)$$

which, for $\lambda = \alpha - 1$, yields

$$\frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n}, \ \alpha \neq 0, \pm 1, \pm 2, \cdots\cdots . \qquad \ldots\ldots (1.2.11)$$

Equations (1.2.7) and (1.2.11) suggest that

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n}, \ n = 1, 2, 3, \cdots\cdots; \lambda \neq 0, \pm 1, \pm 2, \cdots\cdots \qquad \ldots\ldots (1.2.12)$$

and

$$(\lambda)_{m+n} = (\lambda)_m \ (\lambda + m)_n, \qquad \ldots\ldots (1.2.13)$$

which in conjunction with (1.2.12), gives

$$(\lambda)_{n-k} = \frac{(-1)^k (\lambda)_n}{(1-\lambda-n)_k}, \ 0 \leq k \leq n \qquad \ldots\ldots (1.2.14)$$

For $\lambda = 1$, we have

$$(n-k)! = \frac{(-1)^k \ n!}{(-n)_k}, \quad 0 \leq k \leq n, \qquad \ldots\ldots (1.2.15)$$

which may be written as:

$$(-n)_k = \begin{cases} \dfrac{(-1)^k \ n!}{(n-k)!}, & 0 \leq k \leq n, \\ 0, & k > n . \end{cases} \qquad \ldots\ldots (1.2.16)$$

Gauss's Multiplication Theorem

For every positive integer m, we have

$$(\lambda)_{m\,n} = m^{m\,n} \prod_{j=1}^{m} \left(\frac{\lambda + j - 1}{m}\right)_n, \ n = 0, 1, 2, \cdots\cdots \qquad \ldots\ldots (1.2.17)$$

For m = 2, equation (1.2.17) reduces to Legendre's duplication formula

$$(\lambda)_{2n} = 2^{2n} \left(\tfrac{1}{2}\lambda\right)_n \left(\tfrac{1}{2}\lambda + \tfrac{1}{2}\right)_n, \ n=0,1,2,\cdots\cdots \qquad \cdots\cdots (1.2.18)$$

In particular, we have

$$(2n)! = 2^{2n} \left(\tfrac{1}{2}\right)_n \ n! \ and \ (2n+1)! = 2^{2n} \left(\tfrac{3}{2}\right)_n \ n! \ . \qquad \cdots\cdots (1.2.19)$$

The Gaussian Hypergeometric Series

The hypergeometric series given by

$$ {}_2F_1(a,b;c;z) = 1 + \frac{a.b}{1.c} z + \frac{a(a+1)b(b+1)}{1.2.c(c+1)} z^2 + \cdots\cdots$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad , \quad c \neq 0,-1,-2,\cdots\cdots \qquad \cdots\cdots (1.2.20)$$

was introduced by a German mathematician C.F. Gausss (1777-1855). Who in the year (1812) introduced this series into analysis and give the F-notation for it.

The special case a = c and b = 1 or a = 1 and b = c, yield the elementary geometric series.

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots\cdots + z^n + \cdots\cdots \qquad \cdots\cdots (1.2.21)$$

Hence (1.2.20) is called the hypergeometric series or more precisely, Gauss hypergeometric series.

In (1.2.20), $(a)_n$ denotes the pochhammer symbol defined by (1.2.6), z is a real or complex variable, a, b and c are parameters which can take arbitrary real or complex values and $c \neq 0,-1,-2,\cdots\cdots$

If c is zero or negative integer, the series (1.2.20) does not exist and hence the function ${}_2F_1(a,b;c;z)$ is not defined unless one of the parameters a or b is also negative integer such that $-c < -a$ is also negative integer. If either of the parameters a or b is negative integer m, then in this case (1.2.20) reduces to the hypergeometric polynomial defined by

$$ {}_2F_1(-m,b;c;z) = \sum_{n=0}^{\infty} \frac{(-m)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad , \quad (-\infty < z < \infty). \qquad \ldots\ldots (1.2.22) $$

The hypergeometric series (1.2.20) converge absolutely within the unit circle $|z|<1$, provided that $Re\ (c\text{-}a\text{-}b) > 0$ for $z = 1$ and $Re\ (c\text{-}a\text{-}b) > -1$ *for* $z = -1$.

Generalized Hypergeometric Function

A natural generalization of the hypergeometric function ${}_2F_1$ is the generalized hypergeometric function ${}_pF_q$ which is defined as:

$$ {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; \\ b_1,\ldots,b_q; \end{matrix} z\right] = {}_pF_q\left[\begin{matrix} (a); \\ (b); \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n\, z^n}{(b_1)_n \ldots (b_q)_n\, n!} $$

$$ = \sum_{n=0}^{\infty} \frac{((a))_n}{((b))_n} \frac{z^n}{n!} \qquad \ldots\ldots (1.2.23) $$

where $(\lambda)_n$ is the pochhammer symbol defined by (1.2.6)

and $((a))_n = \prod_{j=1}^{p} (a_j)_n$.

Here p and q are positive or zero, the numerator parameters $a_1, \cdots\cdots, a_p$, and the denominators $b_1, \cdots\cdots, b_q$ take on complex values provided that $b_j \neq 0, -1, -2, \cdots\cdots;\ j = 1, 2, \cdots\cdots, q$.

The series in (1.2.23)

(i) Converges for $|z| < \infty$ if $p \le q$,

(ii) Converges for $|z| < 1$ if $p = q + 1$, and

(iii) Diverges for all z, $z \neq 0$, if $p > q + 1$

Furthermore, if we set

$$w = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j ,$$

it is known that the series (1.2.23), with $p = q + 1$, is

(i) Absolutely convergent for $|z| = 1$, if $Re(w) > 0$,

(ii) Conditionally convergent for $|z| = 1$, $z \neq 1$, if $-1 < \mathrm{Re}(w) \le 0$, and

(iii) Divergent for $|z| = 1$, if $Re(w) \le -1$.

An important special cases, when p = q = 1, (1.2.23) reduces to confluent hypergeometric series, ${}_1F_1$ named as Kummer's series [39] and is given by.

$${}_1F_1[a;c;z] = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} = \lim_{|b| \to \infty} {}_2F_1\left[a,b;c;\frac{z}{b}\right] \qquad \ldots\ldots (1.2.24)$$

When p = 2, q = 1, (1.2.23) reduces to the Gauss's hypergeometric function ${}_2F_1$ given by (1.2.20).

1.3 HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

Appell's Functions

In 1880 P. Appell (1855-1930) considered the product of two Gauss functions, viz.

$${}_2F_1(a,b;c;x) \quad {}_2F_1(a',b';c';y)$$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!} \qquad \ldots\ldots (1.3.1)$$

This double series, in itself, yields nothing new, but if one or more of the three pairs of products

$$(a)_m (a')_n, (b)_m (b')_n, (c)_m (c')_n$$

be replaced by the corresponding expressions

$$(a)_{m+n}, (b)_{m+n}, (c)_{m+n},$$

we are led to five distinct possibilities of getting new functions. One such possibility, however, gives us the double series.

$$\sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}$$

which is simply the Gaussian series for

$${}_2F_1(a,b;c;x+y),$$

since it is easily verified that {cf., e.g., [85], p.4)}.

$$\sum_{m,n=0}^{\infty} f(m+n) \frac{x^m}{m!} \frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!}, \qquad \ldots\ldots (1.3.2)$$

or, more generally,

$$\sum_{m_1 \cdots\cdots m_n=0}^{\infty} f(m_1 + \cdots\cdots + m_n) \frac{x_1^{m_1}}{m_1!} \cdots\cdots \frac{x_n^{m_n}}{m_n!}$$

$$= \sum_{m=0}^{\infty} f(m) \frac{(x_1 + \cdots\cdots + x_n)^m}{m!} \qquad \ldots\ldots (1.3.3)$$

The remaining four possibilities led to the four Appell functions of two variables, which are defined below:

$F_1(a,b,b';c;x,y)$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad \ldots\ldots (1.3.4)$$

$\max\{|x|,|y|\} < 1$;

$F_2(a,b,b';c,c';x,y)$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_m(b')_n}{(c)_m\,(c')_n} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad \ldots\ldots (1.3.5)$$

$|x|+|y| < 1$;

$F_3(a,a',b,b';c\,;x,y)$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_m\,(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad \ldots\ldots (1.3.6)$$

$\max\{|x|,|y|\} < 1$;

$F_4(a,b;c,c';x,y)$

$$= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}\,(b)_{m+n}}{(c)_m\,(c')_n} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad \ldots\ldots (1.3.7)$$

$\sqrt{|x|}\,\sqrt{|y|} < 1$.

The functions $\boldsymbol{F_1}$, $\boldsymbol{F_2}$, $\boldsymbol{F_3}$ and $\boldsymbol{F_4}$ given above are all generalization of the Gauss hypergeometric function ${}_2\boldsymbol{F}_1$ given by (1.2.20).

Here, as usual, the denominator parameters c and c' are neither zero nor a negative integer.

The standard work on the theory of Appell series is the monograph by Appell and Kampe` de Fe`riet [4]. See Erdelyi et al [19] for a review of the subsequent work on the subject; see also Bailey [6], Slater {[77], Chapter 8} and Exton {[23], p.23-28}.

Humbert's Functions

In 1920, Humbert [30] gave a list of seven confluent forms of the four Appell's functions and denoted them by $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$.

Here we list two Humbert functions which are use in our subsequent work {see e.g. [19]}.

$$\Phi_3(\beta;\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad \ldots\ldots (1.3.8)$$

$|x| < \infty, |y| < \infty$;

$$\Psi_2(\alpha;\gamma,\gamma';x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m (\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad \ldots\ldots (1.3.9)$$

$|x| < \infty, |y| < \infty$.

Horn's Functions

The efforts of Appell were continued by Horn (1867-1946), who in the year 1931, defined ten hypergeometric functions of two variables and denoted them by $G_1, G_2, G_3, H_1, \cdots\cdots, H_7$. He thus completed the set of all possible complete hypergeometric functions of two variables see {[94], p.56-57} and Erdelyi et al {[19], p.224-228}.

One of them are given below

$$H_3(\alpha,\beta;\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad \text{...... (1.3.10)}$$

$$|x| < r, |y| < s, r+(s-\tfrac{1}{2})^2 = \frac{1}{4}.$$

An interesting result involving Appell's F_2 and Horn's H_3 functions was given by Srivastava {[82], p.681 (2.2)}

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n w^n}{n!} F_2[\alpha,-n,-n;\mu,\lambda;x,y]\, z^n = (1-z)^{\alpha-\lambda}\,\theta^{-\alpha}$$

$$\times H_3\left[\alpha,\mu-\lambda;\mu;\frac{xyz}{\theta^2},\frac{xz}{\theta}\right], \qquad \text{...... (1.3.11)}$$

where $\theta = 1 - z + xz + yz$.

Kampé de Fériet Function

Appell's four double hypergeometric functions F_1, F_2, F_3 *and* F_4, were unified and generalized by Kampé de Fériet {[31], p.401-404} (see also [4], p.150 (29)).

We recall the definition of general double hypergeometric function of Kampé de Fériet in the slightly modified notation of Srivastava and Panda {[95], p.423 (26)}.

$$F^{A:B;D}_{E:G;H}\left[\begin{matrix}(a_A);(b_B);(d_D); \\ (e_E);(g_G);(h_H);\end{matrix}\, x,y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{[(a_A)]_{m+n} [(b_B)]_m [(d_D)]_n}{[(e_E)]_{m+n} [(g_G)]_m [(h_H)]_n} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad \ldots\ldots (1.3.12)$$

where, for convergence,

(i) $A + B < E + G + 1, \quad A + D < E + H + 1, |x| < \infty, |y| < \infty,$ *or*

(ii) $A + B = E + G + 1, \quad A + D = E + H + 1,$ *and*

$$\begin{cases} |x|^{1/(A-E)} \, |y|^{1/(A-E)} < 1, \ if \ A > E, \\ \max \quad \{|x|, |y|\} < 1, \ if \ A \le E. \end{cases} \qquad \ldots\ldots (1.3.13)$$

1.4 HYPERGEOMETRIC FUNCTIONS OF SEVERAL VARIABLES

Lauricella Function of *n* Variables

Lauricella [41] generalized the Appell double hypergeometric functions $F_1, \cdots\cdots, F_4$ (cf. e.g., [19], p.224) to functions of n variables. Two of Lauricella functions, viz. $F_A^{(n)}$ and $F_D^{(n)}$ are defined by

$$F_A^{(n)}(a, b_1, \cdots\cdots, b_n; c_1, \cdots\cdots, c_n; x_1, \cdots\cdots, x_n)$$

$$= \sum_{m_1, \cdots\cdots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots\cdots + m_n} (b_1)_{m_1} \cdots\cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots\cdots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots\cdots \frac{x_n^{m_n}}{m_n!}, \qquad \ldots\ldots (1.4.1)$$

$|x_1| + \cdots\cdots + |x_n| < 1$;

$$F_D^{(n)}(a, b_1, \cdots\cdots, b_n; c; x_1, \cdots\cdots, x_n)$$

$$= \sum_{m_1, \cdots\cdots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots\cdots + m_n} (b_1)_{m_1} \cdots\cdots (b_n)_{m_n}}{(c)_{m_1 + \cdots\cdots + m_n}} \frac{x_1^{m_1}}{m_1!} \cdots\cdots \frac{x_n^{m_n}}{m_n!}, \qquad \ldots\ldots (1.4.2)$$

$\max \{|x_1|, \cdots\cdots, |x_n|\} < 1$.

Clearly, we have

$F_A^{(2)} = F_2 \quad and \; F_D^{(2)} = F_1$.

Lauricella [40] also gave several elementary properties of these functions. A summary of Lauricella's work is given by Appell and Kampé de Fériet ([4], p.114-120). {See also Carlson [13]. Carlitz and Srivastava [12] and Srivastava and Exton [90], [91], [92]}.

Confluent Forms of Lauricella Functions

Two important confluent hypergeometric functions of n variables are the functions $\Phi_2^{(n)}$ and $\Psi_2^{(n)}$ {See e.g. [94] p.62 }.

Here we need $\Psi_2^{(n)}$ only, which defined by

$$\Psi_2^{(n)}\left[a;c_1,\cdots\cdots,c_n;x_1,\cdots\cdots,x_n\right]$$

$$= \sum_{m_1,\cdots\cdots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots\cdots+m_n}}{(c_1)_{m_1}\cdots\cdots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\cdots\frac{x_n^{m_n}}{m_n!}, \qquad \ldots\ldots (1.4.3)$$

clearly, we have

$\Psi_2^{(2)} = \Psi_2$, where Ψ_2 is Humbert confluent hypergeometric function of two variables defined by (1.3.9).

Generalized Lauricella Functions of Several Variables

A further generalization of the Kampé de Fériet Function of two variables $F_{E:G:H}^{A:B;D}$ and Lauricella functions of several variables $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ *and* $F_D^{(n)}$ is due to Srivastava and Daost {cf. [87] and [88]}.

The generalized Lauricella functions of n variable is defined as follows:

$$F^{A:B';\cdots\cdots;B^{(n)}}_{C:D';\cdots\cdots;D^{(n)}}\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \equiv F^{A:B';\cdots\cdots;B^{(n)}}_{C:D';\cdots\cdots;D^{(n)}}\left(\begin{matrix}[(a):(\theta'),\cdots\cdots,\theta^{(n)}]: \\ [(c):\psi',\cdots\cdots,\psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):(\phi')];\cdots\cdots;[(b^{(n)}):(\phi^{(n)})]; \\ [(d'):\delta'];\cdots\cdots;[(d^{(n)}):(\delta^{(n)})];\end{matrix} z_1,\cdots\cdots,z_n\right)$$

$$= \sum_{m_1,\cdots\cdots m_n=0}^{\infty} \Omega(m_1,\cdots\cdots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\cdots\frac{z_n^{m_n}}{m_n!}. \qquad \cdots\cdots (1.4.4)$$

where, for convenience,

$$\Omega(m_1,\cdots\cdots,m_n)$$

$$= \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}} \qquad \cdots\cdots (1.4.5)$$

the coefficients

$$\left\{\begin{matrix}\theta_j^{(k)},j=1,\cdots\cdots,A\ ;\phi_j^{(k)},j=1,\cdots\cdots,B^{(k)}\ ;\psi_j^{(k)},j=1,\cdots\cdots,C\ ; \\ \delta_j^{(k)},j=1,\cdots\cdots,D^{(k)}\ ;\ \forall k\in\{1,\cdots\cdots,n\}\end{matrix}\right.$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\cdots\cdots,a_A$, $(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters.

$$b_j^{(k)},\ j=1,\cdots\cdots,B^{(k)}\ ;\forall k\in\{1,\cdots\cdots,n\},$$

with similar interpretations for and $(d^{(k)})$, $k=1,\cdots\cdots,n$ et cetera.

A detailed discussion of the conditions of convergence of the multiple series (1.4.4) is given in Srivastava and Daoust [89], if the positive constants

$\theta's$, $\psi's, \phi's$ *and* $\delta's$ are all chosen as unity then (1.4.4) reduces to the generalized Kampé de Fériet function given by Karlsson [33] in its more general form.

$$F\begin{smallmatrix} A:B';\cdots\cdots;B^{(n)} \\ C:D';\cdots\cdots;D^{(n)} \end{smallmatrix}[z_1,\cdots\cdots,z_n]$$

$$= F\begin{smallmatrix} A:B';\cdots\cdots;B^{(n)} \\ C:D';\cdots\cdots;D^{(n)} \end{smallmatrix}\left[\begin{matrix}(a):(b');\cdots\cdots;(b^{(n)})\,; \\ (c):(d');\cdots\cdots;(d^{(n)})\,;\end{matrix} z_1,\cdots\cdots,z_n\right]$$

$$= \sum_{m_1,\cdots\cdots,m_n=0}^{\infty} \frac{((a))_{m_1+\cdots\cdots+m_n}\,((b'))_{m_1}\cdots\cdots((b^{(n)}))_{m_n}\; z_1^{m_1}\cdots\cdots z_n^{m_n}}{((c))_{m_1+\cdots\cdots+m_n}\,((d'))_{m_1}\cdots\cdots((d^{(n)}))_{m_n}\; m_1!\cdots\cdots m_n!}. \quad \ldots\ldots(1.4.6)$$

clearly, we have

$$F\begin{smallmatrix} 1:1;\cdots\cdots;1 \\ 0:1;\cdots\cdots;1 \end{smallmatrix} = F_A^{(n)}\ ,\ F\begin{smallmatrix} 0:2;\cdots\cdots;2 \\ 1:0;\cdots\cdots;0 \end{smallmatrix} = F_B^{(n)}\ ,$$

$$F\begin{smallmatrix} 2:0;\cdots\cdots;0 \\ 1:1;\cdots\cdots;1 \end{smallmatrix} = F_C^{(n)}\ \textit{and}\ F\begin{smallmatrix} 1:1;\cdots\cdots;1 \\ 1:0;\cdots\cdots;0 \end{smallmatrix} = F_D^{(n)}\ .$$

The Triple Hypergeometric Functions of Lauricella – Saran

Lauricella {[41], p.114} introduced fourteen complete hypergeometric functions of three variables and of the second order. He denoted his triple hypergeometric functions by the symbols

$F_1, F_2, F_3, \ldots\ldots\ldots, F_{14}$

of which F_1, F_2, F_3, and F_9 corrsepond, respectively, to the three-variable Lauricella functions $F_A^{(3)}, F_B^{(3)}, F_C^{(3)}$ *and* $F_D^{(3)}$ with $n=3$. The remaining ten functions, F_3, F_4, F_6, F_7, F_8, F_{10}, …… , F_{14} of Lauricella's set apparently fell into oblivion [except that there is an isolated appearance of the triple hyypergeometric function F_8 in a paper by Mayr {[46], p.265}. Saran [76] initiated a systematic study of these ten triple hypergeometric functions of Lauricella's set. We give below the definitions of four of these functions using Saran's notations F_G, F_K, F_N and F_S and also indicating Lauricella's notations:

$$F_G(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma_1)_m,(\gamma_2)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \qquad \text{...... (1.4.7)}$$

$|x|<r, |y|<s, |z|<t, \quad r+s=1=r+t$;

$$F_K(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \qquad \text{...... (1.4.8)}$$

$|x|<r, |y|<s, |z|<t, \quad (1-r)(1-s)=t$;

$$F_N(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_1;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_n(\alpha_3)_p(\beta_1)_{m+p}(\beta_2)_n}{(\gamma_1)_m(\gamma_2)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \qquad \text{...... (1.4.9)}$$

$|x|<r, |y|<s, |z|<t, \quad (1-r)s+(1-s)t=0$;

$$F_S(\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_1,\gamma_1;x,y,z)$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma_1)_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \qquad \text{... (1.4.10)}$$

$|x|<r, |y|<s, |z|<t, \quad r+s=rs, \; s=t.$

The General Triple Hypergeometric Series $F^{(3)}$ [x, y, z]

A unification of Lauricella's fourteen-hypergeometric functions $F_1, \cdots\cdots, F_{14}$ of three variables {[41], p.114} and Srivastava's three additional functions H_A, H_B, H_C [81], was introduced by Srivastava (see. e.g. {[80], p.428} and {[94], p.69}) in the form of triple hypergeometric series $F^{(3)}$ $[x, y, z]$ defined as

$$F^{(3)}[x,y,z] = F^{(3)}\left[\begin{matrix} (a)::(b);(b');(b''):(c);(c');(c''); \\ \\ (e)::(g);(g');(g''):(h);(h');(h''); \end{matrix} \; x,y,z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p}\,((b))_{m+n}\,((b'))_{n+p}\,((b''))_{p+m}\,((c))_m\,((c'))_n\,((c''))_p\, x^m y^n z^p}{((e))_{m+n+p}\,((g))_{m+n}\,((g'))_{n+p}\,((g''))_{p+m}\,((h))_m\,((h'))_n\,((h''))_p\, m!n!p!}$$

.... (1.4.11)

where (a) and $((a))$ will mean the sequence of a parameters $a_1, \cdots\cdots, a_A$, and the product $\prod_{j=1}^{A}(a_j)$ respectively.

The triple hypergeometric series (1.4.5) converges absolutely when

$$\begin{cases} 1+E+G+G''+H-A-B-B''-C \geq 0, \\ 1+E+G+G'+H'-A-B-B'-C' \geq 0, \\ 1+E+G'+G''+H''-A-B'-B''-C'' \geq 0, \end{cases} \qquad \text{....... (1.4.12)}$$

where the equalities hold true for suitably constrained values of $|x|, |y|$ *and* $|z|$.

The Multiple Hypergeometric Functions ${}^{(k)}_{(1)}E^{(n)}_{D}$ and ${}^{(k)}_{(2)}E^{(n)}_{D}$

Exton {[23], p. 89 (3.4.1), (3.4.2)} considered the two multiple hypergeometric functions which follow as a generalization of certain of quadrable functions.

The functions are defined as follows:

$$
{}^{(k)}_{(1)}E^{(n)}_{D}\left[a,b_1,\cdots\cdots,b_n;c,c';x_1,\cdots\cdots,x_n\right]
$$

$$
=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots\cdots+m_n}(b_1)_{m_1}\cdots\cdots(b_n)_{m_n}\,x_1^{m_1}\cdots\cdots x_n^{m_n}}{(c)_{m_1+\cdots\cdots+m_k}(c')_{m_{k+1}+\cdots\cdots+m_n}\,m_1!\cdots\cdots m_n!}\,. \qquad \ldots\ldots (1.4.13)
$$

and

$$
{}^{(k)}_{(2)}E^{(n)}_{D}\left[a,a',b_1,\cdots\cdots,b_n;c;\,x_1,\cdots\cdots,x_n\right]
$$

$$
=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots\cdots+m_k}(a')_{m_{k+1}+\cdots\cdots+m_n}(b_1)_{m_1}\cdots\cdots(b_n)_{m_n}\,x_1^{m_1}\cdots\cdots x_n^{m_n}}{(c)_{m_1+\cdots\cdots+m_k}\,m_1!\cdots\cdots m_n!}\,.
$$

$$
\ldots\ldots (1.4.14)
$$

The convergence of the functions (1.4.13) and (1.4.14) are given in {[23], p.91-92}.

We not the following special cases:

$$
{}^{(1)}_{(1)}E^{(2)}_{D}\left[a,b_1,b_2;\,c_1,c_2;x,y\right]
$$

$$
=F_2\left[a,b_1,b_2;c_1,c_2;x,y\right] \qquad \ldots\ldots (1.4.15)
$$

where F_2 is Appell function of two variables defined by (1.3.5)

$$ {}^{(1)}_{(1)}E_D^{(3)}\left[a,b_1,b_2,b_3\,;\,c,c';x,y,z\right] $$

$$ = F_G\left(a,a,a,b_1,b_2,b_3;c,c',c';x,y,z\right) \qquad \ldots\ldots (1.4.16) $$

where F_G is Lauricella-Saran function of three variables

defined by (1.4.7)

$$ {}^{(3)}_{(1)}E_D^{(4)}\left[a,b_1,b_2,b_3,b_4\,;\,c,c';w,x,y,z\right]\,. $$

$$ = K_{11}\left(a,a,a,a,b_1,b_2,b_3,b_4;c,c,c,\,c';w,x,y,z\right). \qquad \ldots\ldots (1.4.17) $$

where K_{11} is Exton's quadriple hypergeometric functions

see {Exton [22] and [23], p.78}.

$$ K_{11}\left(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,\,d;w,x,y,z\right) $$

$$ = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}\,(b_1)_m\,(b_2)_n\,(b_3)_p\,(b_4)_q\,w^m,x^n\,y^p\,z^q}{(c)_{m+n+p}\,(d)_q\;m!n!p!q!}\,. \qquad \ldots\ldots. (1.4.18) $$

$$ {}^{(1)}_{(2)}E_D^{(3)}\left[a,a',b_1,b_2,b_3\,;\,c;x,y,z\right] $$

$$ = F_S\left(a,a',a',b_1,b_2,b_3;c,c\;\;,c;x,y,z\right) \qquad \ldots\ldots (1.4.19) $$

where F_S is the Saran's function of three variables defined by (1.4.10)

$$ {}^{(3)}_{(2)}E_D^{(4)}\left[a,a',b_1,b_2,b_3,b_4\,;\,c;w,x,y,z\right]\,. $$

$$ = K_{15}\left(a,a,a,a',b_1,b_2,b_3,b_4;c,c,c,c;w,x,y,z\right). \qquad \ldots\ldots (1.4.20) $$

where K_{15} is Exton's quadriple hypergeometric functions see {Exton [22] and [23], p.78}

$$ K_{15}\left(a,a,a,b_5,b_1,b_2,b_3,b_4;c,c,c,c;w,x,y,z\right) $$

$$ = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p}\,(b_5)_q(b_1)_m\,(b_2)_n\,(b_3)_p\,(b_4)_q\,w^m\,x^n\,y^p\,z^q}{(c)_{m+n+p+q}\;\;m!n!p!q!}\,. \qquad \ldots\ldots (1.4.21) $$

1.5 ORTHOGONAL POLYNOMIALS

Orthogonal polynomials constitute an important class of special functions in general and of hypergeometric functions in particular.

This class contains many special functions commonly encountered in the applications, e.g., Legendre, Gegenbauer and Jacobi polynomials.

Orthogonal polynomials are of great importance in mathematical physics, approximation theory, the theory of mechanical quadratures, etc.

The subject of orthogonal polynomials is treated in many works such Szego [96], Erdelyi et al [20], Rainville [71], Lebedev [42], Luke [44], McBride [47], Beckmann [8], Askey [5], Danese [18], Carlson [14] and Chihara [16].

Legendre Polynomials

The Legendre polynomials $P_n(x)$ is defined by the generating function.

$$\left(1-2xt+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n , \qquad \text{...... (1.5.1)}$$

in which $\left(1-2xt+t^2\right)^{-1/2}$ denotes the particular branch which tends to 1 as t → 0.

$$\text{Here } P_n(x) = \sum_{n=0}^{[n/2]} \frac{(-1)^k (1/2)_{n-k} (2x)^{n-2k}}{k!(n-2k)!} , \qquad \text{...... (1.5.2)}$$

from which it follows that $P_n(x)$ is a polynomial of degree precisely n in x, from equation (1.5.1), we obtain

$$P_n(-x) = (-1)^n P_n(x)$$

So that $P_n(x)$ is an odd function of x for n odd, an even function of x for n even.

From the relation (1.5.2), we easily find that

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{3}{2}x^2-\frac{1}{2},\ P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x.$$

Rodriques formula of $P_n(x)$ is

$$P_n(x) = \frac{1}{2^n\, n!} D^n (x^2-1)^n \qquad \ldots\ldots (1.5.3)$$

The generating function (1.5.1) can be used to obtain the following hypergeometric form of $P_n(x)$

$$P_n(x) = {}_2F_1\left[\begin{matrix} -n, n+1; \frac{1-x}{2} \\ 1 \quad ; \end{matrix}\right], \qquad \ldots\ldots (1.5.4)$$

or equivalently

$$p_n(x) = (-1)^n\ {}_2F_1\left[\begin{matrix} -n, n+1; \frac{1+x}{2} \\ 1 \quad ; \end{matrix}\right]. \qquad \ldots\ldots (1.5.5)$$

Generalized Laguerre Polynomials

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ is defined by means of generating relation

$$(1-t)^{-(1+\alpha)} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n \qquad \ldots\ldots (1.5.6)$$

where

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n\, x^k}{k!\,(n-k)!(1+\alpha)_k} \qquad \ldots\ldots (1.5.7)$$

The hypergeometric form of generalized Laguerre polynomial given in {[71], p.200 (1)}

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_1F_1\left[-n;\alpha+1;x\right] \qquad \text{...... (1.5.8)}$$

for non-negative integer.

The factor $\frac{(1+\alpha)_n}{n!}$ is inserted for convenience only.

The special case $\alpha = 0$ yields expression of a simple Laguerre polynomial $L_n(x)$.

$$L_n(x) = L_n^{(0)}(x) = {}_1F_1\left[-n;1;x\right] \qquad \text{...... (1.5.9)}$$

From (1.5.7), we obtain

$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = 1 + \alpha - x,$

$L_2^{(\alpha)}(x) = \frac{1}{2}(1+\alpha)\,(2+\alpha)x + \frac{1}{2}x^2.$

Rodrigues formula of $L_n^{(\alpha)}(x)$ is

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}\,e^x}{n!}\,D^n\left[e^{-x}\,x^{n+\alpha}\right]. \qquad \text{...... (1.5.10)}$$

Jacobi Polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ may be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n, 1+\alpha+\beta+n; \frac{1-x}{2}\\ 1+\alpha \qquad ;\end{matrix}\right] \qquad \text{...... (1.5.11)}$$

From (1.5.11) it follows that $P_n^{(\alpha,\beta)}(x)$ is a polynomial of degree precisely n and that

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!} \qquad \text{...... (1.5.12)}$$

Special Cases

(i) When $\alpha = \beta = 0$, the polynomial in (1.5.11) becomes the Legendre polynomial.

$$P_n^{(0,0)}(x) = P_n(x) = {}_2F_1\left[\begin{matrix}-n, n+1; \frac{1-x}{2}\\ 1 \quad ;\end{matrix}\right] \qquad \text{...... (1.5.13)}$$

(ii) if $\alpha = \beta$, the Jacobi polynomial in (1.5.11) reduces to Gegenbauer polynomials

$$C_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n, 1+2\alpha+n; \frac{1-x}{2}\\ 1+\alpha \qquad ;\end{matrix}\right] \qquad \text{...... (1.5.14)}$$

The Laguerre polynomials $L_n^{(\alpha)}(x)$ and the generalized Bessel polynomials $Y_n(a, b; x)$ are, in fact, limiting cases of the Jacobi polynomials {See [94], p.131(1) and [1], p.411 (2)}.

$$L_n^{(\alpha)}(x) = \lim_{|\beta| \to 0} P_n^{(\alpha,\beta)}\left(\frac{1-2x}{\beta}\right) \qquad \text{...... (1.5.15)}$$

$$Y_n\ (a,b;\ x) = \lim_{\beta\to 0} \frac{\Gamma(n+1)\Gamma(\beta)}{\Gamma(\beta+n)} P_n^{(\beta-1,\alpha-\beta-1)}\left(1-\frac{2x\beta}{b}\right) \qquad \text{......} (1.5.16)$$

where the hypergeometric form of the generalized Bessel polynomials $Y_n\ (a,\ b\ ;\ x)$ given in Krall and Frink work [38] see also Grosswald [28] is

$$Y_n\ (a,b;\ x) = {}_2F_0\left[-n, a-1+n; -; \frac{-x}{b}\right] \qquad \text{......} (1.5.17)$$

Jacobi polynomials have many generating functions one of them is Bateman's generating function.

$$\sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x)\,t^n}{(1+\alpha)_n\,(1+\beta)_n} = {}_0F_1\left[-;1+\alpha;\frac{t(x-1)}{2}\right]\ {}_0F_1\left[-;1+\beta;\frac{t(x+1)}{2}\right]$$

$$\text{......} (1.5.18)$$

Equation (1.5.18) was first published by Bateman [7] see also {[71], p.256 (1)}.

From (1.5.18), we note that

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n\, P_n^{(\beta,\alpha)}(x) \qquad \text{......} (1.5.19)$$

Applying (1.5.19) to equation (1.5.11), we obtain the following hypergeometric form of $P_n^{(\alpha,\beta)}(x)$:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n\,(1+\beta)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n, 1+\alpha+\beta+n; & \dfrac{1+x}{2}\\ 1+\beta & ;\end{matrix}\right], \qquad \text{......} (1.5.20)$$

equation (1.5.19) also leads to

$$P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n\,(1+\beta)_n}{n!}. \qquad \text{......} (1.5.21)$$

Rodrigues formula of $P_n^{(\alpha,\beta)}(x)$ is

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n (1-x)^{-\alpha} (1+x)^{-\beta}}{2^n\, n!} D^n \left[(1-x)^{n+\alpha} (1+x)^{n+\beta}\right] \qquad \text{..... (1.5.22)}$$

Generalized Rice Polynomials

Investigation of Rice [71], were continued by Khandeker [37] who in 1964 defined the generalized Rice polynomials as follows:

$$H_n^{(\alpha,\beta)}(\nu,\sigma,x) = \binom{\alpha+n}{n} {}_3F_2 \left[\begin{matrix} -n, \alpha+\beta+n+1, \nu; \\ \alpha+1, \sigma \quad ; \end{matrix} x\right], \qquad \text{...... (1.5.23)}$$

$Re(\alpha) > -1,\ \ Re(\beta) > -1,$

in 1972, Khan {see [34] and [35]} introduced a generalization of Rice polynomial defined by (1.5.23) in the form

$$f_n^{(\alpha,\beta)}\ \left[(a_p)\,,\,(b_q);\, x\right] = \frac{(1+\alpha)_n}{n!}\ {}_{p+2}F_{q+1} \left[\begin{matrix} -n, 1+\alpha+\beta+n,\ (a_p)\ ; \\ \alpha+1, \qquad (b_q)\ ; \end{matrix} x\right],$$

$$\text{...... (1.5.24)}$$

clearly, we note that

$$f_n^{(\alpha,\beta)}(\nu,\sigma,x) = H_n^{(\alpha,\beta)}(\nu,\sigma,x), \qquad \text{...... (1.5.25)}$$

and

$$P_n^{(\alpha,\beta)}(x) = H_n^{(\alpha,\beta)}\left[\nu,\nu,(1-x)/2\right] \qquad \text{...... (1.5.26)}$$

Laguerre Polynomials of Several Variables

In 1991, S.F. Ragab defined Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ as follows:

$$L_n^{(\alpha,\beta)}(x,y)=\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}$$

$$\sum_{r=0}^{n}\frac{(-y)^r L_{n-r}^{(\alpha)}(x)}{r!\Gamma(\alpha+n-r+1)\Gamma(\beta+r+1)} \qquad \ldots\ldots (1.5.27)$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial of one variable defined by (1.5.6).

The definition (1.5.27) is equivalent to the following explicit representation of $L_n^{(\alpha,\beta)}(x,y)$, given by Ragab:

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\,y^r\,x^s}{(\alpha+1)_s(\beta+1)_r\,r!s!} \qquad \ldots\ldots (1.5.28)$$

It may be noted that (1.5.28) may be written as:

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\Psi_2\left[-n;\alpha+1,\beta+1;x,y\right] \qquad \ldots\ldots (1.5.29)$$

where Ψ_2 is a confluent hypergeometric function of two variables defined by (1.3.9).

In 1997, Khan, M.A. and Shukla, A.K. defined the Laguerre polynomials of three-variables and Laguerre polynomial of several variables as follows:

$$L_n^{(\alpha,\beta,\gamma)}(x,y,z)=\frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}{(n!)^3}$$

$$\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s}\frac{(-n)_{r+s+k}\,x^k\,y^s\,z^r}{(\alpha+1)_k(\beta+1)_s(\gamma+1)_r\,r!s!k!}. \qquad \ldots\ldots (1.5.30)$$

In terms of confluent hypergeometric function $\Psi_2^{(3)}$ of three variables defined by (1.4.3), we can write (1.5.30) as

$$L_n^{(\alpha,\beta,\gamma)}(x,y,z)=\frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}{(n!)^3}$$

$$\Psi_2^{(3)}\left[-n;\alpha+1,\beta+1,\gamma+1;x,y,z\right] \qquad \ldots\ldots (1.5.31)$$

and

$$L_n^{(\alpha_1,\cdots\cdots,\alpha_m)}(x_1,\cdots\cdots,x_m)$$

$$=\frac{\prod_{j=1}^{m}(\alpha_j+1)_n}{(n!)^m}\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\cdots\cdots\sum_{r_m=0}^{n-r_1-\cdots\cdots-r_{m-1}}\frac{(-n)_{r_1+\cdots\cdots+r_m}\prod_{j=1}^{m}x_{m+1-j}^{r_j}}{\prod_{j=1}^{m}r_j\prod_{j=1}^{m}(\alpha_j+1)_{m+1-j}} \qquad \ldots\ldots (1.5.32)$$

In terms of confluent hypergeometric function $\Psi_2^{(m)}$ of m variables defined by (1.4.3), we can write (1.5.32) as

$$L_n^{(\alpha_1,\ldots\ldots,\alpha_m)}(x_1,x_2,\cdots\cdots,x_m)$$

$$=\frac{\prod_{j=1}^{m}(\alpha_j+1)_n}{(n!)^m}\Psi_2^{(m)}\left[-n;\ \alpha_1+1,\ \cdots\cdots,\alpha_m+1;x_1,\cdots\cdots,x_m\right]. \qquad \ldots\ldots (1.5.33)$$

Special cases of $L_n^{(\alpha,\beta)}(x,y)$ and $L_n^{(\alpha,\beta,\gamma)}(x,y,z)$

$$\frac{n!}{(\beta+1)_n}L_n^{(\alpha,\beta)}(x,0)=L_n^{(\alpha)}(x) \qquad \ldots\ldots (1.5.34)$$

$$\frac{n!}{(\alpha+1)_n}L_n^{(\alpha,\beta)}(0,y)=L_n^{(\beta)}(y) \qquad \ldots\ldots (1.5.35)$$

$$\frac{n!}{(\gamma+1)_n}L_n^{(\alpha,\beta,\gamma)}(x,y,0)=L_n^{(\alpha,\beta)}(x,y) \qquad \ldots\ldots (1.5.36)$$

$$\frac{n!}{(\beta+1)_n}L_n^{(\alpha,\beta,\gamma)}(x,0,z)=L_n^{(\alpha,\gamma)}(x,z) \qquad \ldots\ldots (1.5.37)$$

$$\frac{n!}{(\alpha+1)_n} L_n^{(\alpha,\beta,\gamma)}(0,y,z) = L_n^{(\beta,\gamma)}(y,z) \qquad \ldots\ldots (1.5.38)$$

$$\frac{(n!)^2}{(\beta+1)_n(\gamma+1)_n} L_n^{(\alpha,\beta,\gamma)}(x,0,0) = L_n^{(\alpha)}(x) \qquad \ldots\ldots (1.5.39)$$

$$\frac{(n!)^2}{(\alpha+1)_n(\gamma+1)_n} L_n^{(\alpha,\beta,\gamma)}(0,y,0) = L_n^{(\beta)}(y) \qquad \ldots\ldots (1.5.40)$$

$$\frac{(n!)^2}{(\alpha+1)_n(\beta+1)_n} L_n^{(\alpha,\beta,\gamma)}(0,0,z) = L_n^{(\gamma)}(z) \qquad \ldots\ldots (1.5.41)$$

1.6 GENERATING FUNCTIONS

The name of generating functions were introduced by Laplace in 1812.Since then the theory of generating functions has been developed into various directions and found wide applications, in various branches of science and technology.

Generating functions may be used to define a set of function, to determine, a differential recurrence relation or a pure recurrence relation, to evaluate certain integrals, et cetera.

Linear Generating Functions

Consider a two-variable function *F* (*x, t*) which possesses a formal (not necessarily convergent for $t \neq 0$) power series expansion in *t* such that

$$F(x,t) = \sum_{r=0}^{\infty} f_n(x)t^n \qquad \ldots\ldots (1.6.1)$$

where each member of the coefficient set $\{f_n(x)\}_{n=0}^{\infty}$ is independent of *t*. Then the expansion (1.6.1) of *F* (*x, t*) is said to have generated the set $\{f_n(x)\}$ and $\{F(x,t)\}$

is called a linear generating function (or, simply, generating function) for the set $\{f_n(x)\}$.

The definition (1.6.1) may be extended slightly to include a generating function of the type:

$$G(x,t) = \sum_{r=0}^{\infty} C_n \, g_n(x) t^n \qquad \ldots\ldots (1.6.2)$$

where the sequence $\{C_n(x)\}_{n=0}^{\infty}$ may contain the parameters of the set $g_n(x)$, but is independent of x and t.

A set of functions may have more than one generating function. However, if $G(x,t) = \sum_{n=0}^{\infty} h_n(x)\, t^n$ then $G(x,t)$ is the unique generator for the set $\{h_n(x)\}$ as the coefficient set.

Bilinear Generating Functions

If a three-variable function $F(x, y, t)$ possesses a formal power series expansion in t such that

$$F(x,y,t) = \sum_{r=0}^{\infty} \gamma_n \, f_n(x) f_n(y) t^n, \qquad \ldots\ldots (1.6.3)$$

where the sequence $\{\gamma_n\}$is independent of x, y and t, then $F(x, y, t)$ is called a bilinear generating function for the set $\{f_n(x)\}$.

More generally, if $F(x, y, t)$ can be expanded in powers of t in the form

$$F(x,y,t) = \sum_{r=0}^{\infty} \gamma_n \, f_{\alpha(n)}(x) f_{\beta(n)}(y) t^n, \qquad \ldots\ldots (1.6.4)$$

where $\alpha(n)$ and $\beta(n)$ are functions of n which are not necessarily equal, we shall still call $F(x, y, t)$ a bilinear generating function for the set $\{f_n(x)\}$.

Bilateral Generating Functions

Suppose that a three-variable function $H(x, y, t)$ has a formal power series expansion in t such that

$$H(x,y,t) = \sum_{r=0}^{\infty} h_n f_n(x) g_n(y) t^n, \qquad \text{...... (1.6.5)}$$

where the sequence $\{h_n\}$ is independent of x, y and t, and the sets of functions $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ are different. Then $H(x, y, t)$ is called a bilateral generating function for the set $\{f_n(x)\}$ or $\{g_n(x)\}$.

The above definition of a bilateral generating function, used earlier by Rainville {[71], p.170} and McBride {[47], p.19}, may be extended to include bilateral generating functions of the type:

$$F(x,y,t) = \sum_{r=0}^{\infty} \gamma_n f_{\alpha(n)}(x) g_{\beta(n)}(y) t^n, \qquad \text{...... (1.6.6)}$$

where the sequence $\{\gamma_n\}$ is independent of x, y and t, the sets of functions $\{f_n(x)\}_{n=0}^{\infty}$ and $\{g_n(x)\}_{n=0}^{\infty}$ are different, and $\alpha(n)$ and $\beta(n)$ are function of n which are not necessarily equal.

Multivariable Generating Functions

In each of the above definitions, the sets generated are functions of only one variable. Suppose now that

$$G(x_1, \cdots\cdots, x_r; t)$$

is a function of $r + 1$ variables, which has a formal expansion in powers of t such that

$$G(x_1, \cdots\cdots, x_r; t) = \sum_{r=0}^{\infty} C_n \, g_n(x_1, \cdots\cdots, x_r) \, t^n, \qquad \ldots\ldots (1.6.7)$$

where the sequence $\{C_n\}$ is independent of the variables $x_1, \cdots\cdots, x_r$ and t. Then we shall say that $G(x_1, \cdots\cdots, x_r; t)$ is a generating function for the set

$$\{g_n(x_1, \cdots\cdots, x_r)\}_{n=0}^{\infty}$$

corresponding to the nonzero coefficients C_n.

It is not difficult to extend the definitions of bilinear and bilateral generating functions to include such multivariable generating functions as

$$F(x_1, \cdots\cdots, x_r; y_1, \cdots\cdots, y_r; t)$$

$$= \sum_{n=0}^{\infty} \gamma_n \, f_{\alpha(n)}(x_1, \cdots\cdots, x_r) f_{\beta(n)}(y_1, \cdots\cdots, y_r; t) \, t^n, \qquad \ldots\ldots (1.6.8)$$

and

$$H(x_1, \cdots\cdots, x_r; y_1, \cdots\cdots, y_s; t)$$

$$= \sum_{n=0}^{\infty} h_n \, f_{\alpha(n)}(x_1, \cdots\cdots, x_r) g_{\beta(n)}(y_1, \cdots\cdots, y_s) \, t^n, \qquad \ldots\ldots (1.6.9)$$

respectively.

Multilinear and Multilateral Generating Functions

A multivariable generating function given by (1.6.7), is said to be a multilinear generating function if

$$g_n(x_1,\cdots\cdots,x_r) = f_{\alpha_1(n)}(x_1)\cdots\cdots g_{\alpha_r(n)}(x_r)\,, \qquad \ldots\ldots (1.6.10)$$

where $\alpha_1(n),\cdots\cdots,\alpha_r(n)$ are functions of n which are not necessarily equal. More generally, if the functions occurring on the right-hand side of (1.6.10) are all different, the multivariable generating function (1.6.7) will be called a multilateral generating function.

Multiple Generating Functions

A natural further extension of the multivariable generating function (1.6.7) is a multiple generating function which may be defined formally by

$$\Psi(x_1,\cdots\cdots,x_r;t_1,\cdots\cdots,t_r)$$

$$= \sum_{n_1,\cdots\cdots,n_r=0}^{\infty} C(n_1,\cdots\cdots,n_r)\,\Gamma_{n_1,\cdots\cdots,n_r}(x_1,\cdots\cdots,x_r)t_1^{n_1},\cdots\cdots,t_r^{n_r}\,, \qquad \ldots\ldots (1.6.11)$$

where the multiple sequence $\{C(n_1,\cdots\cdots,n_r)\}$ is independent of the variable

$x_1,\cdots\cdots,x_r$ and $t_1,\cdots\cdots,t_r$.

CHAPTER-II

On Laguerre Polynomials of Several Variables

CHAPTER-II

ON LAGUERRE POLYNOMIALS OF SEVERAL VARIABLES

2.1 INTRODUCTION

The Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ defined by Ragab {[70] p.253}.

$$L_n^{(\alpha,\beta)}(x,y)=\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-y)^r L_{n-r}^{(\alpha)}(x)}{r!\Gamma(\alpha+n-r+1)\Gamma(\beta+r+1)} \qquad \text{.......(2.1.1)}$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial of one variable defined by (1.5.6).

The definition (2.1.1) is equivalent to the following explicit representation of $L_n^{(\alpha,\beta)}(x,y)$, given by Ragab.

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\, y^r x^s}{(\alpha+1)_s(\beta+1)_r\, r!s!}. \qquad \text{........ (2.1.2)}$$

In terms of confluent hypergeometric function Ψ_2 defined by (1.3.9)

$$\Psi_2(\alpha;\gamma,\gamma';x,y) = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}, \qquad \text{........ (2.1.3)}$$

we can write (2.1.2) as

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\Psi_2[-n;\alpha+1,\beta+1;x,y]. \qquad \text{........ (2.1.4)}$$

Later, the same year Chateerjea [15] gave the following generating function for $L_n^{(\alpha,\beta)}(x,y)$:

$$\sum_{n=0}^{\infty}\frac{n!L_n^{(\alpha,\beta)}(x,y)t^n}{(\alpha+1)_n(\beta+1)_n}=e^t\ {}_0F_1(-;\alpha+1;-xt)\ {}_0F_1(-;\beta+1;-yt) \quad \ldots\ldots. (2.1.5)$$

Chatterjea pointed out that the generating relation (2.1.5) gives the following results:

$$\frac{n!}{(\beta+1)_n}L_n^{(\alpha,\beta)}(x,0)=L_n^{(\alpha)}(x) \quad \ldots\ldots.. (2.1.6)$$

$$\frac{n!}{(\alpha+1)_n}L_n^{(\alpha,\beta)}(0,y)=L_n^{(\beta)}(y) \quad \ldots\ldots.. (2.1.7)$$

which, in particular, yield

$$L_n^{(\alpha,0)}(x,0)=L_n^{(\alpha)}(x) \quad \ldots\ldots.. (2.1.8)$$

$$L_n^{(0,\beta)}(0,y)=L_n^{(\beta)}(y) \quad \ldots\ldots.. (2.1.9)$$

Using the formula

$$\sum_{k=0}^{n}\frac{(pz)^n}{n!(c)_n}\ {}_2F_1(-n,1-c-n;c';q/p)$$

$$={}_0F_1(-;c;pz)\ {}_0F_1(-;c';qz) \quad \ldots\ldots.. (2.1.10)$$

he also obtained the following result from (2.1.5)

$$\frac{(n!)^2L_n^{(\alpha,\beta)}(x,y)}{(\alpha+1)_n(\beta+1)_n}$$

$$=\sum_{r=0}^{n}\frac{(-n)_r x^r}{(\alpha+1)_r r!}\ {}_2F_1(-r,-\alpha-r;\beta+1;y/x) \quad \ldots\ldots. (2.1.11)$$

Further, using $y = x$ and employing the formula

$$ {}_0F_1(-;a;x)\ {}_0F_1(-;b;x) $$

$$ = {}_2F_3\left(\frac{1}{2}a+\frac{1}{2}b,\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2};a,b,a+b-1;4x\right) \qquad \ldots\ldots\ldots (2.1.12) $$

he proved the following result from (2.1.5):

$$ L_n^{(\alpha,\beta)}(x,x)=\frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\ {}_3F_3\left[\begin{matrix} -n,\dfrac{\alpha+\beta+1}{2},\dfrac{\alpha+\beta+2}{2}; \\ \alpha+1,\ \beta+1,\alpha+\beta+1\ ; \end{matrix}\ 4x\right] \ldots\ldots (2.1.13) $$

a mention of which was already made in the work of Ragab.

Lastly, using the following generating function of Bateman

$$ \sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x)t^n}{(1+\alpha)_n\,(1+\beta)_n} $$

$$ = {}_0F_1\left[-;1+\alpha;\frac{t(x-1)}{2}\right]\ {}_0F_1\left[-;1+\beta;\frac{t(x+1)}{2}\right] \qquad \ldots\ldots\ldots(2.1.14) $$

Chatterjea obtained the following result from (2.1.5) on changing $-x$ into ½(x-1) and $-y$ into ½(x+1):

$$ \frac{(n!)\,L_n^{(\alpha,\beta)}\left[\tfrac{1}{2}(1-x),-\tfrac{1}{2}(1+x)\right]}{(\alpha+1)_n\,(\beta+1)_n} $$

$$ \sum_{r=0}^{n}\frac{P_r^{(\alpha,\beta)}(x)}{(n-r)!(1+\alpha)_r\,(1+\beta)_r} \qquad \ldots\ldots\ldots(2.1.15) $$

In 1997 Khan, M.A. and Shukla, A.K. defined the Laguerre polynomials of three variables $L_n^{(\alpha,\beta,\gamma)}(x,y,z)$ and Laguerre polynomials of several variables $L_n^{(\alpha_1,\cdots\cdots,\alpha_m)}(x_1,\cdots\cdots,x_m)$ as follows:

$$L_n^{(\alpha,\beta,\gamma)}(x,y,z)=\frac{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n}{(n!)^3}$$

$$\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=0}^{n-r-s}\frac{(-n)_{r+s+k}\,x^k\,y^s\,z^r}{(\alpha+1)_k\,(\beta+1)_s\,(\gamma+1)_r\,r!s!k!}. \qquad \text{........ (2.1.16)}$$

and

$$L_n^{(\alpha_1,\cdots\cdots,\alpha_m)}(x_1,\cdots\cdots,x_m)\ .$$

$$=\frac{\prod_{j=1}^{m}(\alpha_j+1)_n}{(n!)^m}\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\cdots\cdots\sum_{r_m=0}^{n-r_1-\cdots\cdots-r_{m-1}}\frac{(-n)_{r_1+\cdots\cdots+r_m}\prod_{j=1}^{m}x_{m+1-j}^{r_j}}{\prod_{j=1}^{m}r_j!\prod_{j=1}^{m}(\alpha_j+1)_{m+1-j}} \qquad \text{........ (2.1.17)}$$

In terms of confluent hypergeometric functions $\Psi_2^{(3)}$ and $\Psi_2^{(m)}$ defined by (1.4.3)

$$\Psi_2^{(3)}\left[a;c_1,c_2,c_3;x_1,x_2,x_3\right]$$

$$=\sum_{m_1,m_2,m_3=0}^{\infty}\frac{(a)_{m_1+m_2+m_3}}{(c_1)_{m_1}(c_2)_{m_2}(c_3)_{m_3}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\frac{x_3^{m_3}}{m_3!} \qquad \text{........ (2.1.18)}$$

and

$$\Psi_2^{(m)}\left[a;c_1,\cdots\cdots,c_m;x_1,\cdots\cdots,x_m\right]$$

$$=\sum_{r_1,\cdots\cdots,r_m=0}^{\infty}\frac{(a)_{r_1+\cdots\cdots+r_m}}{(c_1)_{r_1}\cdots\cdots(c_m)_{r_m}}\frac{x_1^{r_1}}{r_1!}\cdots\cdots\frac{x_m^{r_m}}{r_m!}, \qquad \text{........ (2.1.19)}$$

we can write (2.1.16) and (2.1.17) as

$$L_n^{(\alpha,\beta,\gamma)}(x,y,z)=\frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}{(n!)^3}$$

$$\Psi_2^{(3)}\left[-n;\alpha+1,\beta+1,\gamma+1;x,y,z\right]. \qquad \text{........ (2.1.20)}$$

and

$$L_n^{(\alpha_1,\ldots\ldots,\alpha_m)}(x_1,x_2,\cdots\cdots,x_m)$$

$$=\frac{\prod_{j=1}^{m}(\alpha_j+1)_n}{(n!)^m}\Psi_2^{(m)}\left[-n;\alpha_1+1,\cdots\cdots,\alpha_m+1;x_1,\cdots\cdots,x_m\right]. \qquad \text{........ (2.1.21)}$$

Khan and Shukla pointed out that the definition (2.1.20) gives the following particular cases:

$$\frac{n!}{(\gamma+1)_n}L_n^{(\alpha,\beta,\gamma)}(x,y,0)=L_n^{(\alpha,\beta)}(x,y) \qquad \text{........ (2.1.22)}$$

$$\frac{n!}{(\beta+1)_n}L_n^{(\alpha,\beta,\gamma)}(x,0,z)=L_n^{(\alpha,\gamma)}(x,z) \qquad \text{........ (2.1.23)}$$

$$\frac{n!}{(\alpha+1)_n}L_n^{(\alpha,\beta,\gamma)}(0,y,z)=L_n^{(\beta,\gamma)}(y,z) \qquad \text{........ (2.1.24)}$$

$$\frac{(n!)^2}{(\beta+1)_n(\gamma+1)_n}L_n^{(\alpha,\beta,\gamma)}(x,0,0)=L_n^{(\alpha)}(x) \qquad \text{........ (2.1.25)}$$

$$\frac{(n!)^2}{(\alpha+1)_n(\gamma+1)_n} L_n^{(\alpha,\beta,\gamma)}(0,y,0) = L_n^{(\beta)}(y) \qquad \text{........(2.1.26)}$$

$$\frac{(n!)^2}{(\alpha+1)_n(\beta+1)_n} L_n^{(\alpha,\beta,\gamma)}(0,0,z) = L_n^{(\gamma)}(z). \qquad \text{........(2.1.27)}$$

Further, they gave the following generating functions for

$L_n^{(\alpha,\beta)}(x,y)$, $L_n^{(\alpha,\beta,\gamma)}(x,y,z)$ and $L_n^{(\alpha_1,\ldots\ldots,\alpha_m)}(x_1,x_2,\cdots\cdots,x_m)$

$$\sum_{n=0}^{\infty}\frac{n!(\lambda)_n L_n^{(\alpha,\beta)}(x,y)t^n}{(\alpha+1)_n(\beta+1)_n}$$

$$=(1-t)^{-\lambda}\ \Psi_2\left[\lambda;\alpha+1,\beta+1;\frac{xt}{t-1},\frac{yt}{t-1}\right], \qquad \text{........(2.1.28)}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2 L_n^{(\alpha,\beta,\gamma)}(x,y,z)t^n}{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}$$

$$=e^t\ {}_0F_1(-;\alpha+1;-xt)\ {}_0F_1(-;\beta+1;-yt)_0\ F_1(-;\gamma+1;-zt) \qquad \text{.......(2.1.29)}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^2(\lambda)_n t^n}{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n} L_n^{(\alpha,\beta,\gamma)}(x,y,z)$$

$$=(1-t)^{-\lambda}\ \Psi_2^{(3)}\left[\lambda;\ \alpha+1,\beta+1,\gamma+1;\ \frac{xt}{t-1},\frac{yt}{t-1},\frac{zt}{t-1}\right] \qquad \text{........(2.1.30)}$$

$$\sum_{n=0}^{\infty}\frac{(n!)^{m-1}\ t^n}{\prod_{j=1}^{m}(\alpha_j+1)_n} L_n^{(\alpha_1,\cdots\cdots,\alpha_m)}(x_1,\cdots\cdots x_m)$$

$$=e^t\prod_{j=1}^{m}{}_0F_1(-;\alpha_j+1;-x_jt) \qquad \text{.......(2.1.31)}$$

and

$$\sum_{n=0}^{\infty}\frac{(n!)^{m-1}(\lambda)_n t^n}{\prod_{j=1}^{m}(\alpha_j+1)_n} L_n^{(\alpha_1,\cdots\cdots,\alpha_m)}(x_1,\cdots\cdots x_m)$$

$$=(1-t)^{-\lambda}\,\Psi_2^{(m)}\left[\lambda;\alpha_1+1,\cdots\cdots,\alpha_m+1;\frac{x_1 t}{t-1},\cdots\cdots,\frac{x_m t}{t-1}\right]. \qquad \ldots\ldots (2.1.32)$$

The main purpose of this chapter is to introduce the above generating functions (2.1.28)-(2.1.32) to establish some finite sum properties, transformations and generating functions for the Laguerre polynomials of two, three, four and several variables. Also many generating functions involving generalized hypergeometric function ${}_pF_q$, confluent hypergeometric function of several variables $\Psi_2^{(m)}$ and generalized Kampé de Fériet function of several variables $F^{A:B';\cdots\cdots;B^{(n)}}_{C:D';\cdots\cdots;D^{(n)}}[x_1,\cdots\cdots,x_n]$ defined by (1.2.23), (1.4.3) and (1.4.6) are shown here to be special cases of our main results.

2.2 FINITE SUM PROPERTIES AND TRANSFORMATIONS OF $L_n^{(\alpha,\beta,\gamma)}(x,y,z)$

Following generating function for Laguerre polynomials of three variables given by Khan, M.A and Shukla, A.K. {[36], p.162 (20)}:

$$\sum_{n=0}^{\infty}\frac{(n!)^2\,L_n^{(\alpha,\beta,\gamma)}(x,y,z)t^n}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n}$$

$$=e^t\;{}_0F_1(-;\alpha+1\,;-xt)\;{}_0F_1(-;\beta+1\,;-yt)\;{}_0F_1(-;\gamma+1\,;-zt)\quad\ldots\ldots.\ (2.2.1)$$

Now, on changing $-y$ into $\frac{1}{2}(y-1)$ and $-z$ into $\frac{1}{2}(y+1)$ and using the results {[71], p.201 (1)} and {[71], p. 256 (1)}.

$$\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)t^n}{(\alpha+1)_n}=e^t\;{}_0F_1(-;\alpha+1\,;-xt)\qquad\ldots\ldots.(2.2.2)$$

and

$$\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x)t^n}{(1+\alpha)_n\,(1+\beta)_n}$$

$$={}_0F_1\left[-;1+\alpha;\frac{t(x-1)}{2}\right]\;{}_0F_1\left[-;1+\beta;\frac{t(x+1)}{2}\right],\qquad\ldots\ldots\ (2.2.3)$$

we have

$$\sum_{n=0}^{\infty}\frac{(n!)^2\,L_n^{(\alpha,\beta,\gamma)}\left[x,\frac{1}{2}(1-y),-\frac{1}{2}(1+y)\right]t^n}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n}$$

$$=\left(\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)t^n}{(\alpha+1)_n}\right)\left(\sum_{k=0}^{\infty}\frac{P_k^{(\beta,\gamma)}(y)t^k}{(1+\beta)_k\,(1+\gamma)_k}\right)\qquad\ldots\ldots\ (2.2.4)$$

Comparing the coefficient of t^n on both sides of (2.2.4), we get

$$L_n^{(\alpha,\beta,\gamma)}\left[x,\frac{1}{2}(1-y),-\frac{1}{2}(1+y)\right]$$

$$=\frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}{(n!)^2}\sum_{k=0}^{n}\frac{L_{n-k}^{(\alpha)}(x)P_k^{(\beta,\gamma)}(y)}{(\alpha+1)_{n-k}(\beta+1)_k\,(\gamma+1)_k}\qquad\ldots\ldots\ (2.2.5)$$

Next, if in (2.2.1), we use the result {[15], p.263 (2)}

$$\sum_{n=0}^{\infty} \frac{n!\, L_n^{(\alpha,\beta)}(x,y)\, t^n}{(\alpha+1)_n\,(\beta+1)_n}$$

$$= e^t \;{}_0F_1(-;\alpha+1\,;-xt)\;{}_0F_1(-;\beta+1\,;-yt), \qquad \ldots\ldots\ldots (2.2.6)$$

we get

$$\frac{(n!)^2\, L_n^{(\alpha,\beta,\gamma)}(x,y,z)}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n}$$

$$= \sum_{k=0}^{n} \frac{(n-k)!\, L_{n-k}^{(\alpha,\beta)}(x,y)\,(-z)^k}{(\alpha+1)_{n-k}\,(\beta+1)_{n-k}\,(\gamma+1)_k\, k!} \qquad \ldots\ldots\ldots (2.2.7)$$

Now, employing the result (2.2.2) and the result {[19], p.186]}

$$\sum_{k=0}^{n} \frac{(pz)^n}{n!(c)_n}\;{}_2F_1(-n,1-c-n;c';q/p)$$

$$= {}_0F_1(-;c;pz)\;{}_0F_1(-;c';qz) \qquad \ldots\ldots\ldots (2.2.8)$$

to (2.2.1), we get

$$\frac{(n!)^2\, L_n^{(\alpha,\beta,\gamma)}(x,y,z)}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n}$$

$$= \sum_{k=0}^{n} \frac{L_{n-k}^{(\alpha)}(x)\,(-y)^k}{(\alpha+1)_{n-k}\,(1+\beta)_{n-k}\, k!}\;{}_2F_1(-k,-\beta-k;\gamma+1;z/y) \qquad \ldots\ldots\ldots (2.2.9)$$

In (2.2.9), if we put $z=y$ and use Gauss's theorem and Legendre's duplication formula

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c-a-b)\,\Gamma(c)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad \ldots\ldots(2.2.10)$$

and

$$(\lambda)_{2n} = 2^{2n}\left(\tfrac{1}{2}\lambda\right)_n\left(\tfrac{1}{2}\lambda+\tfrac{1}{2}\right)_n,\; n=0,1,2,\cdots\cdots \qquad \ldots\ldots(2.2.11)$$

we obtain

$$L_n^{(\alpha,\beta,\gamma)}(x,y,y) = \frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}{(n!)^2}$$

$$\sum_{k=0}^{n} \frac{\left(\frac{\beta+\gamma+2}{2}\right)_k \left(\frac{\beta+\gamma+1}{2}\right)_k (-4y)^k}{k!(1+\alpha)_{n-k}(1+\beta)_k(1+\gamma)_k(\beta+\gamma+1)_k} \quad L_{n-k}^{(\alpha)}(x)$$

$$\ldots.. \; (2.2.12)$$

which in view of the definition {[71], p. 200 (1)}

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \, {}_1F_1\left[\begin{matrix} -n\,; \\ \alpha+1; \end{matrix} x\right], \qquad \ldots\ldots. \; (2.2.13)$$

can be put in the following form:

$$L_n^{(\alpha,\beta,\gamma)}(x,y,y) = \frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}{(n!)^2}$$

$$\sum_{k=0}^{n} \frac{\left(\frac{\beta+\gamma+2}{2}\right)_k \left(\frac{\beta+\gamma+1}{2}\right)_k (-4y)^k}{k!(n-k)!(1+\alpha)_{n-k}(1+\beta)_k(1+\gamma)_k(\beta+\gamma+1)_k} \; {}_1F_1\left[\begin{matrix} -n+k\,; \\ \alpha+1\;; \end{matrix} x\right] \ldots \; (2.2.14)$$

Now by expanding ${}_1F_1(.)$ in the series form and adjusting the parameters, we arrive to the result

$$L_n^{(\alpha,\beta,\gamma)}(x,y,y) = \frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}{(n!)^3}$$

$$F_{0:1;3}^{1:0;2}\left[\begin{matrix} -n: \; - \;; \frac{\beta+\gamma+2}{2}, \frac{\beta+\gamma+1}{2}; \\ -:\alpha+1; \beta+1, \gamma+1, \beta+\gamma+1; \end{matrix} x, 4y\right] \qquad \ldots. \; (2.2.15)$$

where $F^{A:B;B'}_{C:D;D'}[x,y]$ is the Kampé de Fériet function of two variables defined by (1.3.12).

Now, we consider the following result {[36], p.160 (5.3)}

$$L_n^{(\alpha,\beta,\gamma)}(x,y,z)=\frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}{(n!)^3}$$

$$\Psi_2^{(3)}\left[-n;\alpha+1,\beta+1,\gamma+1;x,y,z\right]. \qquad \text{.....}(2.2.16)$$

where $\Psi_2^{(3)}$ is a confluent hypergeometric function of three variables defined by (1.4.3).

clearly from (2.2.15) and (2.2.16), we get

$$\Psi_2^{(3)}\left[-n;\alpha+1,\beta+1,\gamma+1;x,y,y\right].$$

$$=F^{1:0;2}_{0:1;3}\left[\begin{matrix}-n: & - & ;\frac{\beta+\gamma+2}{2},\frac{\beta+\gamma+1}{2}; \\ -: & \alpha+1; & \beta+1,\gamma+1,\beta+\gamma+1;\end{matrix}\; x,4y\right]. \qquad \text{.....}(2.2.17)$$

On the other-hand if we apply the result {[36], p. 157 (2.1)}

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\Psi_2\left[-n;\alpha+1,\beta+1;x,y\right]. \qquad \text{.....}(2.2.18)$$

we derive from (2.2.16) the following result:

$$L_n^{(\alpha,\beta,\gamma)}(x,y,z)=\frac{(\alpha+1)_n(\beta+1)_n(\gamma+1)_n}{(n!)^3}$$

$$\sum_{k=0}^{n}\frac{(-n)_k x^k[(n-k)!]^2}{(\alpha+1)_k(\beta+1)_{n-k}(\gamma+1)_{n-k}k!}L_{n-k}^{(\beta,\gamma)}(y,z) \qquad \text{...}(2.2.19)$$

2.3 FINITE SUM PROPERTIES AND TRANSFORMATIONS OF $L_n^{(\alpha,\beta,\gamma,\delta)}(w,x,y,z)$

We consider the generating function {[36], p.163 (7.4)} for m=4

$$\sum_{n=0}^{\infty}\frac{(n!)^3\,L_n^{(\alpha,\beta,\gamma,\delta)}(w,x,y,z)t^n}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n\,(\delta+1)_n}$$

$$= e^t\,{}_0F_1(-;\alpha+1;-wt)\,{}_0F_1(-;\beta+1;-xt)\,{}_0F_1(-;\gamma+1;-yt)\,{}_0F_1(-;\delta+1;-zt)$$

.....(2.3.1)

Now, on changing $-y$ into $\frac{1}{2}(z-1)$ and $-z$ into $\frac{1}{2}(z+1)$ and using the results (2.2.6) and (2.2.3), we get

$$\sum_{n=0}^{\infty}\frac{(n!)^3\,L_n^{(\alpha,\beta,\gamma,\delta)}\left[w,x,\frac{1}{2}(1-z),-\frac{1}{2}(1+z)\right]t^n}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n\,(\delta+1)_n}$$

$$=\left(\sum_{n=0}^{\infty}\frac{n!\,L_n^{(\alpha,\beta)}(w,x)t^n}{(\alpha+1)_n\,(\beta+1)_n}\right)\left(\sum_{k=0}^{\infty}\frac{P_k^{(\gamma,\delta)}(z)t^k}{(1+\gamma)_k\,(1+\delta)_k}\right) \qquad \text{..... (2.3.2)}$$

comparing the coefficient of t^n on both sides of (2.3.2), we get

$$L_n^{(\alpha,\beta,\gamma,\delta)}\left[w,x,\frac{1}{2}(1-z),-\frac{1}{2}(1+z)\right]=\frac{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n\,(\delta+1)_n}{(n!)^3}$$

$$\sum_{k=0}^{n}\frac{(n-k)!\,L_{n-k}^{(\alpha,\beta)}(w,x)P_k^{(\gamma,\delta)}(z)}{(1+\alpha)_{n-k}(1+\beta)_{n-k}\,(1+\gamma)_k\,(1+\delta)_k}$$

..... (2.3.3)

Next, using (2.2.1) to (2.3.1), we get

$$\frac{(n!)^3\, L_n^{(\alpha,\beta,\gamma,\delta)}(w,x,y,z)}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n\,(\delta+1)_n}$$

$$= \sum_{k=0}^{n} \frac{[(n-k)!]^2\, L_{n-k}^{(\alpha,\beta,\gamma)}(w,x,y)\,(-z)^k}{(\alpha+1)_{n-k}\,(\beta+1)_{n-k}\,(\gamma+1)_{n-k}\,(\delta+1)_k\, k!} \qquad \text{..... (2.3.4)}$$

Now, if we apply (2.2.6) and (2.2.8) to (2.3.1), we get

$$\frac{(n!)^3\, L_n^{(\alpha,\beta,\gamma,\delta)}(w,x,y,z)}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n\,(\delta+1)_n}$$

$$= \sum_{p=0}^{n} \frac{(n-p)!\, L_{n-p}^{(\alpha,\beta)}(w,x)\,(-y)^p}{(\alpha+1)_{n-p}\,(\beta+1)_{n-p}\,(\gamma+1)_p\, p!}\ {}_2F_1(-p,-\gamma-p;\delta+1;z/y) \qquad \text{..... (2.3.5)}$$

which for z=y, reduces to

$$\frac{(n!)^3\, L_n^{(\alpha,\beta,\gamma,\delta)}(w,x,z,z)}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n\,(\delta+1)_n}$$

$$= \sum_{p=0}^{n} \frac{(n-p)!\left(\frac{\gamma+\delta+2}{2}\right)_p \left(\frac{\gamma+\delta+1}{2}\right)_p (-4z)^p}{p!(1+\alpha)_{n-p}(1+\beta)_{n-p}(1+\gamma)_p(1+\delta)_p(\gamma+\delta+1)_p}\ L_{n-p}^{(\alpha,\beta)}(w,x)$$

..... (2.3.6)

using $w=x$ in (2.3.6) and employing the formula {[70], p.262 (19)}

$$L_n^{(\alpha,\beta)}(x,x) = \frac{(1+\alpha)_n\,(1+\beta)_n}{(n!)^2}\ {}_3F_3\left[\begin{matrix} -n, \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}; \\ \alpha+1,\ \beta+1, \alpha+\beta+1\ ; \end{matrix}\ 4x\right], \qquad \text{..... (2.3.7)}$$

we get

$$\frac{(n!)^3\, L_n^{(\alpha,\beta,\gamma,\delta)}(x,x,z,z)}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n\,(\delta+1)_n}$$

$$=\sum_{p=0}^{n}\frac{\left(\frac{\gamma+\delta+2}{2}\right)_p\left(\frac{\gamma+\delta+1}{2}\right)_p(-4z)^p}{p!(n-p)!(1+\gamma)_p(1+\delta)_p(\gamma+\delta+1)_p}\ {}_3F_3\left[\begin{array}{l}-n+p,\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+1}{2};\\ \alpha+1,\ \beta+1,\alpha+\beta+1\quad;\end{array}4x\right]$$

..... (2.3.8)

Now, by expanding ${}_3F_3(.)$ in the series form and adjusting the parameters, we get

$$\frac{(n!)^4\, L_n^{(\alpha,\beta,\gamma,\delta)}(x,x,z,z)}{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n\,(\delta+1)_n}$$

$$=F^{1:2;2}_{0:3;3}\left[\begin{array}{l}-n:\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+1}{2};\frac{\gamma+\delta+2}{2},\frac{\gamma+\delta+1}{2};\\ -:\ \alpha+1,\beta+1,\alpha+\beta+1;\gamma+1,\delta+1,\gamma+\delta+1;\end{array}4x,4z\right].$$

..... (2.3.9)

Our result (2.3.9) together with the result {[36], p. 163 (7.3)}

$$L_n^{(\alpha_1,\,\ldots\ldots,\,\alpha_m)}(x_1,x_2,\cdots\cdots,x_m)$$

$$=\frac{\prod_{j=1}^{m}(\alpha_j+1)_n}{(n!)^m}\ \Psi_2^{(m)}\left[-n;\alpha_1+1,\cdots\cdots,\alpha_m+1;x_1,\cdots\cdots,x_m\right],\ m=4$$

..... (2.3.10)

yields

$$\Psi_2^{(4)}\left[-n;\ \alpha+1,\ \beta+1,\gamma+1,\delta+1;x,x,z,z\right]$$

$$= F_{0:3;3}^{1:2;2}\left[\begin{array}{l} -n:\dfrac{\alpha+\beta+2}{2},\dfrac{\alpha+\beta+1}{2};\dfrac{\gamma+\delta+2}{2},\dfrac{\gamma+\delta+1}{2};4x,4z \\ -:\ \alpha+1,\beta+1,\alpha+\beta+1;\gamma+1,\delta+1,\gamma+\delta+1; \end{array}\right] \quad (2.3.11)$$

Yet, another results for $L_n^{(\alpha,\beta,\gamma,\delta)}(w,x,y,z)$ as follows:

$$L_n^{(\alpha,\beta,\gamma,\delta)}(w,x,y,z) = \frac{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n\,(\delta+1)_n}{(n!)^4}$$

$$\sum_{p,q=0}^{n}\frac{(-n)_{p+q}\,w^p x^q\left[(n-p-q)!\right]^2}{(1+\alpha)_p(1+\beta)_q\,(1+\gamma)_{n-p-q}\,(1+\delta)_{n-p-q}\,p!q!}\ L_{n-p-q}^{(\gamma,\delta)}(y,z) \qquad ....(2.3.12)$$

and

$$L_n^{(\alpha,\beta,\gamma,\delta)}(w,x,y,z) = \frac{(\alpha+1)_n\,(\beta+1)_n\,(\gamma+1)_n\,(\delta+1)_n}{(n!)^4}$$

$$\sum_{p=0}^{n}\frac{(-n)_p\,w^p\left[(n-p)!\right]^3}{(1+\alpha)_p(1+\beta)_{n-p}\,(1+\gamma)_{n-p}\,(1+\delta)_{n-p}\,p!}\ L_{n-p}^{(\beta,\gamma,\delta)}(x,y,z).$$

.....(2.3.13)

2.4 FINITE SUM PROPERTIES AND TRANSFORMATIONS OF $\mathbf{L}_n^{(\alpha_1,\ldots,\alpha_s)}(x_1,\cdots,x_s)$

On the same lines of derivation of a finite sum properties of $L_n^{(\alpha,\beta,\gamma)}(x,y,z)$ and $L_n^{(\alpha,\beta,\gamma,\delta)}(w,x,y,z)$, we have the following results for $L_n^{(\alpha_1,\ldots,\alpha_s)}(x_1,\cdots\cdots,x_s)$:

$$\frac{(n!)^{s-1}\, L_n^{(\alpha_1,\ldots\ldots,\alpha_s)}\left[x_1,\ldots,x_{s-2},\frac{1}{2}(1-x_s),-\frac{1}{2}(1+x_s)\right]}{\prod_{j=1}^{s}(1+\alpha_j)_n}$$

$$=\sum_{r=0}^{n}\frac{(n-r)!\,L_{n-r}^{(\alpha_1,\ldots\ldots,\alpha_{s-2})}(x_1,\cdots\cdots,x_{s-2})\;\;P_r^{(\alpha_{s-1},\alpha_s)}(x_s)}{\prod_{j-1}^{s-2}(1+\alpha_j)_{n-r}\;\;(1+\alpha_{s-1})_r(1+\alpha_s)_r}, \qquad \ldots\ldots (2.4.1)$$

$$\frac{(n!)^{s-1}\, L_n^{(\alpha_1,\ldots\ldots,\alpha_s)}\left[x_1,\ldots,x_s\right]}{\prod_{j=1}^{s}(1+\alpha_j)_n}$$

$$=\sum_{r=0}^{n}\frac{[(n-r)!]^{s-2}\, L_{n-r}^{(\alpha_1,\ldots\ldots,\alpha_{s-1})}\;(x_1,\cdots\cdots,x_{s-1})(-x_s)^r}{\prod_{j-1}^{s-1}(1+\alpha_j)_{n-r}(1+\alpha_s)_r\, r!}, \qquad \ldots\ldots (2.4.2)$$

$$L_n^{(\alpha_1,\ldots\ldots,\alpha_s)}\left[x_1,x_2,x_2,\cdots\cdots,x_{(s+1)/2},x_{(s+1)/2}\right]=\frac{\prod_{j=1}^{s}(1+\alpha_j)_n}{(n!)^s}$$

$$\times F\begin{matrix}1:0;2;\cdots\cdots;\,2^{[(s+1)/2]}\\ 0:1;\,3;\cdots\cdots;\,3^{[(s+1)/2]}\end{matrix}\left[\begin{matrix}-\;\;n:\;-\;\;;\;\dfrac{2+\alpha_2+\alpha_3}{2},\;\dfrac{1+\alpha_2+\alpha_3}{2};\cdots\cdots;\\ -\;\;:1+\alpha_1\,;1+\alpha_2,1+\alpha_3,1+\alpha_2+\alpha_3\,;\;\;\cdots\cdots;\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots\cdots;\;\dfrac{2+\alpha_{s-1}+\alpha_s}{2},\dfrac{1+\alpha_{s-1}+\alpha_s}{2};\\ \cdots\cdots;\;1+\alpha_{s-1},1+\alpha_s,1+\alpha_{s-1}+\alpha_s\,;\end{matrix}\;x_1,4x_2,\cdots,4x_{(s+1)/2}\right], \text{if } s=3,5,7\ldots\ldots$$

$$\ldots\ldots(2.4.3)$$

and

$$L_n^{(\alpha_1,\ldots\ldots,\alpha_s)}\left[x_1,x_1,x_2,x_2,\cdots\cdots,x_{s/2},x_{s/2}\right]=\frac{\prod_{j=1}^{s}(1+\alpha_j)_n}{(n!)^s}$$

$$\times F\begin{matrix}1:2;\cdots\cdots;2^{(s/2)}\\0:3;\cdots\cdots;3^{(s/2)}\end{matrix}\left[\begin{matrix}-\ n & : & \dfrac{2+\alpha_1+\alpha_2}{2},\dfrac{1+\alpha_1+\alpha_2}{2};\cdots\cdots;\\ - & : & 1+\alpha_1,1+\alpha_2,\ 1+\alpha_1+\alpha_2;\ \cdots\cdots;\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots\cdots;\ \dfrac{2+\alpha_{s-1}+\alpha_s}{2},\dfrac{1+\alpha_{s-1}+\alpha_s}{2};\\ \cdots\cdots;\ 1+\alpha_{s-1},1+\alpha_s,1+\alpha_{s-1}+\alpha_s;\end{matrix}4x_1,4x_2,\cdots,4x_{s/2}\right],if,s=4,6,8,\ldots\ldots$$

..... (2.4.4)

where $F\begin{matrix}A:B';\cdots\cdots;B^{(n)}\\C:D';\cdots\cdots;D^{(n)}\end{matrix}[x_1,\cdots\cdots,x_n]$ is the generalized Kampé de Fériet function of several variables defined by (1.4.6). Our results (2.4.3) and (2.4.4) together with the definition {[36], p.163 (7.3).

$$L_n^{(\alpha_1,\ldots\ldots,\alpha_m)}(x_1,x_2,\cdots\cdots,x_m)$$

$$=\frac{\prod_{j=1}^{m}(\alpha_j+1)_n}{(n!)^m}\Psi_2^{(m)}[-n;\alpha_1+1,\cdots\cdots,\alpha_m+1;x_1,\cdots\cdots,x_m]. \qquad ..... (2.4.5)$$

yields

$$\Psi_2^{(s)}[-n;\alpha_1+1,\cdots\cdots,\alpha_s+1;x_1,x_2,x_2,\cdots\cdots,x_{(s+1)/2},x_{(s+1)/2}].$$

$$=F\begin{matrix}1:0;2;\cdots\cdots;2^{[(s+1)/2]}\\0:1;3;\cdots\cdots;3^{[(s+1)/2]}\end{matrix}\left[\begin{matrix}-\ n:- & ;\dfrac{2+\alpha_2+\alpha_3}{2},\dfrac{1+\alpha_2+\alpha_3}{2};\cdots\cdots;\\ - & :1+\alpha_1;1+\alpha_2,1+\alpha_3,1+\alpha_2+\alpha_3;\ \cdots\cdots;\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots\cdots;\ \dfrac{2+\alpha_{s-1}+\alpha_s}{2},\dfrac{1+\alpha_{s-1}+\alpha_s}{2};\\ \cdots\cdots;\ 1+\alpha_{s-1},1+\alpha_s,1+\alpha_{s-1}+\alpha_s;\end{matrix}x_1,4x_2,\cdots,4x_{(s+1)/2}\right],if\ s=3,5,7\ldots\ldots$$

..... (2.4.6)

and

$$\Psi_2^{(s)}\left[-n;\ \alpha_1+1,\cdots\cdots,\alpha_s+1;x_1,x_1,x_2,x_2,\cdots\cdots,x_{s/2},x_{s/2}\right].$$

$$= F\begin{matrix}1:2;\cdots\cdots;\ 2^{(s/2)}\\ 0:3;\cdots\cdots;\ 3^{(s/2)}\end{matrix}\left[\begin{matrix}-\ n & : & \dfrac{2+\alpha_1+\alpha_2}{2},\ \dfrac{1+\alpha_1+\alpha_2}{2};\cdots\cdots; \\ - & : & 1+\alpha_1,1+\alpha_2,\ 1+\alpha_1+\alpha_2;\ \ \cdots\cdots;\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots\cdots;\ \dfrac{2+\alpha_{s-1}+\alpha_s}{2},\dfrac{1+\alpha_{s-1}+\alpha_s}{2}; \\ \cdots\cdots;\ 1+\alpha_{s-1},1+\alpha_s,1+\alpha_{s-1}+\alpha_s;\end{matrix} 4x_1,4x_2,\cdots,4x_{s/2}\right], if\ \ s=4,6,8,\ldots\ldots \qquad \ldots..\ (2.4.7)$$

where $\Psi_2^{(s)}$ is a confluent hypergeometric function of s-variables defined by (1.4.3).

Further, we mention the following results:

$$L_n^{(\alpha_1,\ldots\ldots,\alpha_s)}(x_1,\cdots\cdots,x_s)=\frac{\prod\limits_{j=1}^{s}(1+\alpha_j)_n}{(n!)^s}$$

$$\sum_{r_1,\ldots,r_{(s-1)/2}=0}^{n}\frac{(-n)_{r_1+\ldots\ldots+r_{(s-1)/2}}\prod\limits_{j=1}^{(s-1)/2}x_j^{r_j}\left[(n-r_1-\ldots\ldots\ldots-r_{(s-1)/2})!\right]^{(s+1)/2}}{\prod\limits_{j=1}^{(s-1)/2}(1+\alpha_j)_{r_j}\prod\limits_{j=(s+1)/2}^{s}(1+\alpha_j)_{n-r_1-\ldots\ldots\ldots-r_{(s-1)/2}}\prod\limits_{j=1}^{(s-1)/2}r_j!}$$

$$L_{n-r_1-\ldots\ldots\ldots-r_{(s-1)/2}}^{(\alpha_{(s+1)/2},\ldots\ldots,\alpha_s)}\left[x_{(s+1)/2},\ldots\ldots\ldots,x_s\right],\ \text{if s}=3,5,7,\ldots\ldots\ . \qquad \ldots..\ (2.4.8)$$

and

$$L_n^{(\alpha_1,\ldots\ldots,\alpha_s)}(x_1,\cdots\cdots,x_s)=\frac{\prod\limits_{j=1}^{s}(1+\alpha_j)_n}{(n!)^s}$$

$$\sum_{r_1,\ldots,r_{s/2}=0}^{n}\frac{(-n)_{r_1+\ldots\ldots+r_{s/2}}\prod\limits_{j=1}^{s/2}x_j^{r_j}\left[(n-r_1-\ldots\ldots\ldots-r_{s/2})!\right]^{s/2}}{\prod\limits_{j=1}^{s/2}(1+\alpha_j)_{r_j}\prod\limits_{j=(s+2)/2}^{s}(1+\alpha_j)_{n-r_1-\ldots\ldots\ldots-r_{s/2}}\prod\limits_{j=1}^{s/2}r_j!}$$

$$L_{n-r_1-\ldots\ldots\ldots-r_{s/2}}^{(\alpha_{(s+2)/2},\ldots\ldots,\alpha_s)}\left[x_{(s+2)/2},\ldots\ldots\ldots,x_s\right],\ \text{if s}=4,6,8,\ldots\ldots\ . \qquad \ldots.(2.4.9)$$

2.5 GENERATING FUNCTIONS INVOLVING $L_n^{(\alpha,\beta)}(x,y)$

In what follows, we have established two generating functions. First is a double generating function for the product of a pair of Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ and the second is a bilateral generating function involving $L_n^{(\alpha,\beta)}(x,y)$.

$$\sum_{m,n=0}^{\infty} \frac{(\lambda)_{m+n}\, t^m\, (-t)^n\, m!\, n!}{(\alpha+1)_m\, (\beta+1)_m\, (\gamma+1)_n\, (\delta+1)_n}\, L_m^{(\alpha,\beta)}(x,y)\, L_n^{(\gamma,\delta)}(z,w)$$

$$= \Psi_2^{(4)}\left[\lambda;\alpha+1,\beta+1,\gamma+1,\delta+1;-xt,-yt,zt,wt\right] \qquad \text{.....} (2.5.1)$$

and

$$\sum_{n=0}^{\infty} \frac{n!\,(\lambda)_n}{(\alpha+1)_n\,(\beta+1)_n}\, \Psi_2\left[\lambda+n;\gamma,\delta;u,v\right] L_n^{(\alpha,\beta)}(x,y)\, t^n$$

$$= (1-t)^{-\lambda}\, \Psi_2^{(4)}\left[\lambda;\gamma,\delta,\alpha+1,\beta+1;\frac{u}{1-t},\frac{v}{1-t},\frac{xt}{t-1},\frac{yt}{t-1}\right]. \qquad \text{........} (2.5.2)$$

Proof of (2.5.1) and (2.5.2):

In the proofs of the above two generating functions, we will use the relation

$$(\lambda)_{m+n} = (\lambda)_n\, (\lambda+n)_m \qquad \text{........} (2.5.3)$$

In order to prove (2.5.1), let us write A for the terms on the L.H.S. of (2.5.1). Using (2.1.28) and (2.5.3), we get

$$A=\sum_{n=0}^{\infty}\frac{(\lambda)_n(-t)^n\,n!\,L_n^{(\gamma,\delta)}(z,w)}{(\gamma+1)_n(\delta+1)_n}(1-t)^{-(\lambda+n)}\,\Psi_2\left[\lambda+n;\alpha+1,\beta+1;\frac{xt}{t-1},\frac{yt}{t-1}\right]$$

.... (2.5.4)

Now, expressing Ψ_2 in series form and using again (2.1.28) and (2.5.3), we get

$$A=(1-t)^{-\lambda}\sum_{p,q=0}^{\infty}\frac{(\lambda)_{p+q}\left(\frac{-xt}{1-t}\right)^p\left(\frac{-yt}{1-t}\right)^q}{(\alpha+1)_p(\beta+1)_q\,p!\,q!}$$

$$\times(1-t)^{\lambda+p+q}\,\Psi_2\left[\lambda+p+q;\gamma+1,\delta+1;zt,wt\right]$$ .... (2.5.5)

Next, expressing Ψ_2 in series and adjusting the parameters, we shall arrive the required result. To prove (2.5.2), we adopt the same method as used in the proof of (2.5.1) we use (2.1.28) and (2.5.3).

Now we mention some interesting special cases of the generating functions (2.5.1) and (2.5.2).

On putting $x = y$ and $z = w$, in (2.5.1), we get

$$\sum_{m,n=0}^{\infty}\frac{(\lambda)_{m+n}\,t^m(-t)^n\,m!\,n!}{(\alpha+1)_m(\beta+1)_m(\gamma+1)_n(\delta+1)_n}\,L_m^{(\alpha,\beta)}(x,x)\,L_n^{(\gamma,\delta)}(z,z)$$

$$=F^{1:2;2}_{0:3;3}\left[\begin{array}{l}\lambda:\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta+2}{2};\frac{\gamma+\delta+1}{2},\frac{\gamma+\delta+2}{2};\\ -:\alpha+1,\beta+1,\alpha+\beta+1;\gamma+1,\delta+1,\gamma+\delta+1;\end{array}-4xt,4zt\right],$$

.... (2.5.6)

where $F^{A:B;B'}_{C:D;D'}\left[x,y\right]$ is the Kampé de Fériet function of two variables defined by (1.3.12).

For $y = w = 0$, (2.5.1) reduces to

$$\sum_{m,n=0}^{\infty} \frac{(\lambda)_{m+n}\, t^m\, (-t)^n}{(\alpha+1)_m\, (\gamma+1)_n}\, L_m^{(\alpha)}(x)\, L_n^{(\gamma)}(z)$$

$$= \Psi_2\,(\lambda;\alpha+1,\gamma+1;-xt,zt), \qquad \text{........ (2.5.7)}$$

which for $\alpha=\gamma$, $\lambda=\alpha+1$ and using the result {[93], p.322 (182)}.

$$\Psi_2\,(\gamma;\gamma,\gamma;x,y)=e^{x+y}\ {}_0F_1\,(-;\gamma;xy) \qquad \text{........ (2.5.8)}$$

reduces to a known result due to Exton {[25], p.7 (4.9)}

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, t^m\, (-t)^n}{(\alpha+1)_m\, (\alpha+1)_n}\, L_m^{(\alpha)}(x)\, L_n^{(\alpha)}(z)$$

$$= \exp(tz-tx)\ {}_0F_1\left[-;\alpha+1\,;-t^2xz\right]. \qquad \text{........ (2.5.9)}$$

And for $z=\gamma=0$, reduces to another known result {[94], p.132 (5)}.

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m\, t^m}{(\alpha+1)_m}\, L_m^{(\alpha)}(x)$$

$$= (1-t)^{-\lambda}\ {}_1F_1\left[\lambda;\alpha+1\,;\frac{xt}{t-1}\right]. \qquad \text{........ (2.5.10)}$$

Now, on putting u = v, y=0, in (2.5.2), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\alpha+1)_n} {}_3F_3\left[\lambda+n, \frac{\gamma+\delta}{2}, \frac{\gamma+\delta-1}{2}; \gamma, \delta, \gamma+\delta-1; 4u\right] L_n^{(\alpha)}(x) t^n$$

$$= (1-t)^{-\lambda} F^{1:2;0}_{0:3;1}\left[\begin{matrix} \lambda: \frac{\gamma+\delta}{2}, \frac{\gamma+\delta-1}{2}; & - & ; \\ -: \quad \gamma, \delta, \gamma+\delta-1; & \alpha+1; \end{matrix} \frac{4u}{1-t}, \frac{xt}{t-1}\right] \quad \ldots\ldots\ldots (2.5.11)$$

If we put $u=v, x=y, \beta=\alpha+1, \gamma=\delta$, in (2.5.2), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_2\left[\begin{matrix} \lambda+n, \frac{2\gamma-1}{2}; \\ \gamma \quad , 2\gamma-1; \end{matrix} 4u\right] {}_2F_2\left[\begin{matrix} -n, \quad \frac{2\alpha+3}{2}; \\ \alpha+2, 2\alpha+2; \end{matrix} 4x\right] t^n$$

$$= (1-t)^{-\lambda} F^{1:1;1}_{0:2;2}\left[\begin{matrix} \lambda: \quad \frac{2\gamma-1}{2}; & \frac{2\alpha+3}{2}; \\ -: \gamma, 2\gamma-1; & \alpha+2, 2\alpha+2; \end{matrix} \frac{4u}{1-t}, \frac{4xt}{t-1}\right]. \quad \ldots\ldots\ldots (2.5.12)$$

2.6 GENERATING FUNCTIONS INVOLVING $L_n^{(\alpha,\beta,\gamma)}$ (x,y,z).

In this section we shall generalize the relations (2.5.1) and (2.5.2) of section (2.5) and we will use the same analysis to obtain the following generating functions involving $L_n^{(\alpha,\beta,\gamma)}(x,y,z)$.

$$\sum_{m,n=0}^{\infty} \frac{(\lambda)_{m+n} t^m (-t)^n (m!)^2 (n!)^2 L_m^{(a,b,c)}(x,y,z)\, L_n^{(d,e,f)}(u,v,w)}{(a+1)_m (b+1)_m (c+1)_m (d+1)_n (e+1)_n (f+1)_n}$$

$$= \Psi_2^{(6)}\left[\lambda; a+1, b+1, c+1, d+1, e+1, f+1; -xt, -yt, -zt, ut, vt, wt\right]$$

$$\ldots\ldots\ldots (2.6.1)$$

and

$$\sum_{n=0}^{\infty} \frac{(n!)^2 (\lambda)_n}{(\alpha+1)_n (\beta+1)_n (\gamma+1)_n} \Psi_2^{(3)}\left[\lambda+n; \mu, \delta, \rho; u, v, w\right] L_n^{(\alpha,\beta,\gamma)}(x,y,z)\, t^n$$

$$= (1-t)^{-\lambda}\ \Psi_2^{(6)}\left[\lambda;\mu,\delta,\rho,\alpha+1,\beta+1,\gamma+1;\ \frac{u}{1-t},\frac{v}{1-t},\frac{w}{1-t},\frac{xt}{t-1},\frac{yt}{t-1},\frac{zt}{t-1}\right]$$

.... (2.6.2)

Particular cases of our results (2.6.1) and (2.6.2) are as follows:

If in (2.6.1), we put $x = y$, $z = -u$ and $v = w$, we get

$$\sum_{m,n=0}^{\infty} \frac{(\lambda)_{m+n}\, t^m (-t)^n (m!)^2 (n!)^2}{(a+1)_m (b+1)_m (c+1)_m (d+1)_n (e+1)_n (f+1)_n} L_m^{(a,b,c)}(x,x,-u)\ L_n^{(d,e,f)}(u,w,w)$$

$$= F\begin{matrix}1:2;2;2\\0:3;3;3\end{matrix}\left[\begin{matrix}\lambda:\frac{a+b}{2},\ \frac{a+b-1}{2};\frac{c+d}{2},\frac{c+d-1}{2};\frac{e+f}{2},\frac{e+f-1}{2};\\ -:a,b,\ a+b-1;\ c,d,c+d-1;\ e,f,e+f-1;\end{matrix} -4xt,4ut,4wt\right]$$

.... (2.6.3)

where $F^{A:B';\cdots\cdots;B^{(n)}}_{C:D';\cdots\cdots;D^{(n)}}[x_1,\cdots\cdots,x_n]$ is the generalized Kampé de Fériet function of n-variables defined by {[24], p.28 (1.4.3)}.

$$F\begin{matrix}A:B';\cdots\cdots;B^{(n)}\\C:D';\cdots\cdots;D^{(n)}\end{matrix}[x_1,\cdots\cdots,x_n]\ .$$

$$= F\begin{matrix}A:B';\cdots\cdots;B^{(n)}\\C:D';\cdots\cdots;D^{(n)}\end{matrix}\left[\begin{matrix}(a):(b);\cdots\cdots;(b^{(n)});\\(c):(d');\cdots\cdots;(d^{(n)});\end{matrix}x_1,\cdots\cdots,x_n\right]$$

$$= \sum_{m_1,\cdots\cdots,m_n=0}^{\infty} \frac{((a))_{m_1+\cdots\cdots+m_n}((b'))_{m_1}\ \cdots\cdots\ ((b^{(n)}))_{m_n}\ x_1^{m_1}\cdots\cdots x_n^{m_n}}{((c))_{m_1+\cdots\cdots+m_n}((d'))_{m_1}\ \cdots\cdots\ ((d^{(n)}))_{m_n}\ m_1!\cdots\cdots m_n!},$$

..... (2.6.4)

where $((a))_m = \prod_{j=1}^{A}(a_j)_m$.

For x = w = 0, (2.6.3) evidently reduces to

$$\sum_{m,n=0}^{\infty} \frac{(\lambda)_{m+n}\, t^m (-t)^n}{(c+1)_m\,(d+1)_n}\, L_m^{(c)}(-u)\, L_n^{(d)}(u)$$

$$= {}_3F_3\left[\lambda, \frac{c+d}{2}, \frac{c+d-1}{2}; c, d, c+d-1; 4ut\right]. \qquad \text{........ (2.6.5)}$$

Now, if in (2.6.2), we put $u = v = w$ and $z = 0$, then we get

$$\sum_{n=0}^{\infty} \frac{n!(\lambda)_n}{(\alpha+1)_n(\beta+1)_n}\, F^{1:0;2}_{0:1;3}\left[\begin{matrix} \lambda+n:-; \dfrac{\delta+\rho}{2}, \dfrac{\delta+\rho-1}{2}; u, 4u \\ -:\mu; \delta, \rho, \quad \delta+\rho-1; \end{matrix}\right] L_n^{(\alpha,\beta)}(x,y)\, t^n$$

$$= (1-t)^{-\lambda}\, F^{1:0;0;0;2}_{0:1;1;1;3}\left[\begin{matrix} \lambda: - ; - ; - ; \dfrac{\delta+\rho}{2}, \dfrac{\delta+\rho-1}{2}; \\ -:\alpha+1; \beta+1; \mu; \delta, \rho, \delta+\rho-1; \end{matrix} \frac{xt}{t-1}, \frac{yt}{t-1}, \frac{u}{1-t}, \frac{4u}{1-t}\right]$$

.... (2.6.6)

which for $y = 0$ reduces to

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\alpha+1)_n}\, F^{1:0;2}_{0:1;3}\left[\begin{matrix} \lambda+n:-; \dfrac{\delta+\rho}{2}, \dfrac{\delta+\rho-1}{2}; u, 4u \\ - :\mu; \delta, \rho, \quad \delta+\rho-1; \end{matrix}\right] L_n^{(\alpha)}(x)\, t^n$$

$$= (1-t)^{-\lambda}\, F^{1:0;0;2}_{0:1;1;3}\left[\begin{matrix} \lambda: - ; - ; \dfrac{\delta+\rho}{2}, \dfrac{\delta+\rho-1}{2}; \\ -:\alpha+1; \mu; \delta, \rho, \delta+\rho-1; \end{matrix} \frac{xt}{t-1}, \frac{u}{1-t}, \frac{4u}{1-t}\right]$$

.... (2.6.7)

Again in (2.6.2), putting $u = v = w$ and $x = y = z$, we then get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{1:0;2}_{0:1;3}\left[\begin{array}{l}-n:\ -\ ;\frac{\beta+\gamma+2}{2},\frac{\beta+\gamma+1}{2};\\ -:\alpha+1;\beta+1,\gamma+1,\beta+\gamma+1;\end{array}x,4x\right]$$

$$\times F^{1:0;2}_{0:1;3}\left[\begin{array}{l}\lambda+n\ :-;\ \frac{\delta+\rho}{2},\ \frac{\delta+\rho-1}{2};\\ -\quad :\mu;\ \delta\ ,\rho,\delta+\rho-1;\end{array}u,4u\right]t^n=(1-t)^{-\lambda}$$

$$F^{1:0;0;2;2}_{0:1;1;3;3}\left[\begin{array}{l}\lambda\ :-;\ -\ ;\frac{\delta+\rho}{2},\frac{\delta+\rho-1}{2};\ \frac{\beta+\gamma+2}{2},\frac{\beta+\gamma+1}{2};\\ -:\mu;\alpha+1;\delta,\rho,\ \delta+\rho-1;\ \beta+1,\gamma+1,\beta+\gamma+1;\end{array}\right.$$

$$\left.\frac{u}{1-t},\frac{xt}{t-1},\frac{4u}{1-t},\frac{4xt}{t-1}\right]. \quad \ldots\ldots..(2.6.8)$$

2.7 GENERATING FUNCTIONS INVOLVING $\mathbf{L}_n^{(\alpha_1,\cdots\cdots,\alpha_s)}(\mathbf{x}_1,\cdots\cdots,\mathbf{x}_s)$

On the same line of derivation of the generating functions for $L_n^{(\alpha,\beta)}(x,y)$ and $L_n^{(\alpha,\beta,\gamma)}(x,y,z)$, it is easy to derive the following two generating functions for the Laguerre polynomials of s-variables.

$$\sum_{m,n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n(m!)^{s-1}(n!)^{s-1}}{\prod\limits_{j=1}^{s}(1+\alpha_j)_m\ \prod\limits_{j=1}^{s}(1+\gamma_j)_n}L_m^{(\alpha_1,\cdots\cdots,\alpha_s)}(x_1,\cdots\cdots,x_s)\,L_n^{(\gamma_1,\cdots\cdots,\gamma_s)}(z_1,\cdots\cdots,z_s)$$

$$=\Psi_2^{(2s)}[\lambda;1+\alpha_1,\cdots\cdots,1+\alpha_s,1+\gamma_1,\cdots\cdots,1+\gamma_s;-x_1t,\cdots\cdots,-x_st,z_1t,\cdots\cdots,z_st]\ldots\ldots..(2.7.1)$$

and

$$\sum_{n=0}^{\infty}\frac{(n!)^{s-1}(\lambda)_n}{\prod\limits_{j=1}^{s}(1+\alpha_j)_n}\Psi_2^{(s)}\left[\lambda+n;\beta_1,\cdots\cdots,\beta_s;\nu_1,\cdots\cdots,\nu_S\right]L_n^{(\alpha_1,\cdots\cdots,\alpha_s)}(x_1,\cdots\cdots,x_s)t^n$$

$$=(1-t)^{-\lambda}\ \Psi_2^{(2s)}\left[\lambda;\beta_1,\cdots,\beta_s,1+\alpha_1,\cdots,1+\alpha_s;\frac{\nu_1}{1-t},\cdots,\frac{\nu_s}{1-t},\frac{x_1t}{t-1},\cdots\frac{x_st}{t-1}\right].$$

$$\ldots..(2.7.2)$$

On other hand, as a particular case of our result (2.7.1), if we setting $x_1=\cdots\cdots=x_s=-x,\ z_1=\cdots\cdots=z_s=z$, we obtain the following results:

Case I: If $s=2,4,6,\cdots\cdots$, we get

$$\sum_{m,n=0}^{\infty}\frac{(\lambda)_{m+n}\,t^m\,(-t)^n\,(m!)^{s-1}(n!)^{s-1}}{\prod_{j=1}^{s}(1+\alpha_j)_m\,\prod_{j=1}^{s}(1+\gamma_j)_n}\,L_m^{(\alpha_1,\cdots\cdots,\alpha_s)}(-x,\cdots\cdots,-x)L_n^{(\gamma_1,\cdots\cdots,\gamma_s)}(z,\cdots\cdots,z)$$

$$=F\begin{array}{l}1:2;\ \cdots\cdots,2^{(s)}\\0:3;\cdots\cdots,3^{(s)}\end{array}\left[\begin{array}{l}\lambda:\dfrac{2+\alpha_1+\alpha_2}{2},\dfrac{1+\alpha_1+\alpha_2}{2};\ \cdots\cdots;\dfrac{2+\alpha_{s-1}+\alpha_s}{2},\dfrac{1+\alpha_{s-1}+\alpha_s}{2};\\ -:1+\alpha_1,1+\alpha_2,1+\alpha_1+\alpha_2;\ \cdots\cdots;1+\alpha_{s-1},1+\alpha_s,1+\alpha_{s-1}+\alpha_s;\end{array}\right.$$

$$\left.\begin{array}{l}\dfrac{2+\gamma_1+\gamma_2}{2},\dfrac{1+\gamma_1+\gamma_2}{2};\cdots;\dfrac{2+\gamma_{s-1}+\gamma_s}{2},\dfrac{1+\gamma_{s-1}+\gamma_s}{2};\\ 1+\gamma_1,1+\gamma_2,1+\gamma_1+\gamma_2;\cdots;1+\gamma_{s-1},1+\gamma_s,1+\gamma_{s-1}+\gamma_s;\end{array}4xt,\cdots,4xt,4zt,\cdots,4zt\right].$$

.... (2.7.3)

Case II:

If $s=1,3,5,\cdots\cdots$ we get

$$\sum_{m,n=0}^{\infty}\frac{(\lambda)_{m+n}\,t^m\,(-t)^n\,(m!)^{s-1}(n!)^{s-1}}{\prod_{j=1}^{s}(1+\alpha_j)_m\,\prod_{j=1}^{s}(1+\gamma_j)_n}\,L_m^{(\alpha_1,\cdots\cdots,\alpha_s)}(-x,\cdots\cdots,-x)\,L_m^{(\gamma_1,\cdots\cdots,\gamma_s)}(z,\cdots\cdots,z)$$

$$=F\begin{array}{l}1:0;0;2;\cdots\cdots;2;2^{(s+1)}\\0:1;1;3;\cdots\cdots;3;3^{(s+1)}\end{array}\left[\begin{array}{l}\lambda:\quad -;\ -;\ \dfrac{2+\alpha_2+\alpha_3}{2},\dfrac{1+\alpha_2+\alpha_3}{2};\cdots\cdots;\\ -:1+\alpha_1;1+\gamma_1;1+\alpha_2,1+\alpha_3,1+\alpha_2+\alpha_3;\cdots\cdots;\end{array}\right.$$

$$\begin{array}{l}\dfrac{2+\alpha_{s-1}+\alpha_s}{2},\dfrac{1+\alpha_{s-1}+\alpha_s}{2};\dfrac{2+\gamma_2+\gamma_3}{2},\dfrac{1+\gamma_2+\gamma_3}{2};\cdots\cdots;\\ 1+\alpha_{s-1},1+\alpha_s,1+\alpha_{s-1}+\alpha_s;\ 1+\gamma_2,1+\gamma_3,1+\gamma_2+\gamma_3;\cdots\cdots;\end{array}$$

$$\left.\begin{array}{l}\dfrac{2+\gamma_{s-1}+\gamma_s}{2},\dfrac{1+\gamma_{s-1}+\gamma_s}{2};\\ 1+\gamma_{s-1},1+\gamma_s,1+\gamma_{s-1}+\gamma_s;\end{array}xt,zt,4xt,\cdots\cdots,4xt,4zt,\cdots\cdots,4zt\right].$$

...... (2.7.4)

Finally, it may be of interest to remark that the generating function (2.7.2) written as:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\Psi_2^{(s)}\left[\lambda+n;\beta_1,\cdots\cdots,\beta_s;v_1,\cdots\cdots,v_s\right]$$

$$\times\Psi_2^{(s)}\left[-n;1+\alpha_1,\cdots\cdots,1+\alpha_s;x_1,\cdots\cdots,x_s\right]t^n$$

$$=(1-t)^{-\lambda}\;\Psi_2^{(2s)}\left[\lambda;\beta_1,\cdots,\beta_s,1+\alpha_1,\cdots,1+\alpha_s;\frac{v_1}{1-t},\cdots,\frac{v_s}{1-t},\frac{x_1t}{t-1},\cdots,\frac{x_st}{t-1}\right]$$

.... (2.7.5)

Now on putting $v_1=\cdots\cdots=v_s=v$ and $x_1=\cdots\cdots=x_s=x$, we have the following results:

Case I:

If $s=4,6,8,\ ....$, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F\begin{matrix}1:2;\ \cdots;2^{(s/2)}\\0:3;\cdots;\ 3^{(s/2)}\end{matrix}\left[\begin{matrix}\lambda+n:\dfrac{\beta_1+\beta_2}{2},\dfrac{\beta_1+\beta_2-1}{2};\cdots\\-:\beta_1,\beta_2,\ \ \beta_1+\beta_2-1;\cdots\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots;\dfrac{\beta_{s-1}+\beta_s}{2},\dfrac{\beta_{s-1}+\beta_s-1}{2};\\\cdots;\ \beta_{s-1},\beta_s,\ \beta_{s-1}+\beta_s-1;\end{matrix}\ 4v,\cdots,4v\right]$$

$$\times F\begin{matrix}1:2;\ \cdots;2^{(s/2)}\\0:3;\cdots;\ 3^{(s/2)}\end{matrix}\left[\begin{matrix}-n:\dfrac{2+\alpha_1+\alpha_2}{2},\dfrac{1+\alpha_1+\alpha_2}{2};\cdots\\-:1+\alpha_1,1+\alpha_2,\ 1+\alpha_1+\alpha_2;\cdots\end{matrix}\right.$$

$$\left.\begin{matrix}\cdots;\ \dfrac{2+\alpha_{s-1}+\alpha_s}{2},\dfrac{1+\alpha_{s-1}+\alpha_s}{2};\\\cdots;\ 1+\alpha_{s-1},1+\alpha_s,1+\alpha_{s-1}+\alpha_s;\end{matrix}\ 4x,\cdots,4x\right]$$

$$=(1-t)^{-\lambda}\,F^{1:2;\;\cdots\cdots;2^{(s)}}_{0:3;\;\cdots\cdots;\,3^{(s)}}\left[\begin{array}{l}\lambda:\frac{\beta_1+\beta_2}{2},\frac{\beta_1+\beta_2-1}{2};\cdots\cdots;\frac{\beta_{s-1}+\beta_s}{2},\frac{\beta_{s-1}+\beta_s-1}{2};\\ -:\;\beta_1,\beta_2,\;\beta_1+\beta_2-1;\cdots\cdots;\;\beta_{s-1},\beta_s,\;\beta_{s-1}+\beta_s-1;\end{array}\right.$$

$$\begin{array}{l}\frac{2+\alpha_1+\alpha_2}{2},\frac{1+\alpha_1+\alpha_2}{2};\cdots\cdots;\;\frac{2+\alpha_{s-1}+\alpha_s}{2},\frac{1+\alpha_{s-1}+\alpha_s}{2};\\ 1+\alpha_1,1+\alpha_2,1+\alpha_1+\alpha_2;\cdots\cdots;\;1+\alpha_{s-1},1+\alpha_s,1+\alpha_{s-1}+\alpha_s;\end{array}$$

$$\left.\frac{4v}{1-t},\cdots\cdots,\frac{4v}{1-t},\frac{4xt}{t-1},\cdots\cdots,\frac{4xt}{t-1}\right]. \qquad \ldots\ldots\ldots(2.7.6)$$

Case II:

If $s=3,5,7,\cdots\cdots$, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\;F^{1:0;2;\;\cdots;2^{[(s+1)/2]}}_{0:1;3;\;\cdots;\;3^{[(s+1)/2]}}\left[\begin{array}{l}\lambda+n:-;\frac{\beta_2+\beta_3}{2},\frac{\beta_2+\beta_3-1}{2};\cdots\\ -:\beta_1;\beta_2,\beta_3,\;\beta_2+\beta_3-1;\cdots\end{array}\right.$$

$$\left.\begin{array}{l}\cdots;\frac{\beta_{s-1}+\beta_s}{2},\frac{\beta_{s-1}+\beta_s-1}{2};\\ \cdots;\;\beta_{s-1},\beta_s,\;\beta_{s-1}+\beta_s-1;\end{array}v,4v\cdots,4v\right]$$

$$\times\,F^{1:0;2;\cdots\cdots;\;2^{[(s+1)/2]}}_{0:1;\;3;\cdots\cdots;\;3^{[(s+1)/2]}}\left[\begin{array}{l}-\;\;n:\;-\;\;;\frac{2+\alpha_2+\alpha_3}{2},\;\;\frac{1+\alpha_2+\alpha_3}{2};\cdots\cdots;\\ -\;\;:1+\alpha_1;1+\alpha_2,1+\alpha_3,1+\alpha_2+\alpha_3;\;\;\cdots\cdots;\end{array}\right.$$

$$\left.\begin{array}{l}\frac{2+\alpha_{s-1}+\alpha_s}{2},\frac{1+\alpha_{s-1}+\alpha_s}{2};\\ 1+\alpha_{s-1},1+\alpha_s,1+\alpha_{s-1}+\alpha_s;\end{array}x,4x,\cdots,4x\right]$$

$$=(1-t)^{-\lambda}\;F^{1:0;0;2;\;\cdots\cdots;2^{(s+1)}}_{0:1;1;3;\;\cdots\cdots;\,3^{(s+1)}}\left[\begin{array}{l}\lambda:-\;;\;-\;\;;\frac{\beta_2+\beta_3}{2},\frac{\beta_2+\beta_3-1}{2};\cdots\\ -:\beta_1;1+\alpha_1;\;\beta_2,\beta_3,\;\beta_2+\beta_3-1;\cdots\end{array}\right.$$

$$\begin{matrix} \cdots; \dfrac{\beta_{s-1}+\beta_s}{2}, \dfrac{\beta_{s-1}+\beta_s-1}{2}; \dfrac{2+\alpha_2+\alpha_3}{2}, \dfrac{1+\alpha_2+\alpha_3}{2}; \cdots\cdots; \\ \cdots; \beta_{s-1}, \beta_s, \beta_{s-1}+\beta_s-1; 1+\alpha_2, 1+\alpha_3, 1+\alpha_2+\alpha_3; \cdots\cdots; \end{matrix}$$

$$\left.\begin{matrix} \dfrac{2+\alpha_{s-1}+\alpha_s}{2}, \dfrac{1+\alpha_{s-1}+\alpha_s}{2}; \\ 1+\alpha_{s-1}, 1+\alpha_s, 1+\alpha_{s-1}+\alpha_s; \end{matrix} \dfrac{v}{1-t}, \dfrac{xt}{t-1}, \dfrac{4v}{1-t}, \cdots\cdots, \dfrac{4v}{1-t}, \dfrac{4xt}{t-1}, \cdots\cdots, \dfrac{4xt}{t-1}\right].$$

…….. (2.7.7)

CHAPTER-III

On Double Generating Relations Involving Certain Hypergeometric Functions

CHAPTER-III

ON DOUBLE GENERATING RELATIONS INVOLVING CERTAIN HYPERGEOMETRIC FUNCTIONS

3.1 INTRODUCTION

The main object of the present chapter is to obtain some double generating relations involving Gauss's hypergeometric function ${}_2F_1$, generalized hypergeometric function ${}_pF_q$, and generalized Kampé de Fériet functions of several variables $F^{A:B';\cdots\cdots;B^{(n)}}_{C:D';\cdots\cdots;D^{(n)}}[x_1, \cdots\cdots, x_n]$ defined by (1.2.20), (1.2.23) and (1.4.6) respectively.

A generating functions given by Exton {[26], p.403 (2.2)} and {[25], p.12 (6.12)} for the confluent hypergeometric function ${}_1F_1$ play a key role in obtaining these generating relations by application of Laplace and inverse Laplace transform.

Further double generating relations involving Appell's function F_4, Legendre polynomial $P_n(x)$, Laguerre polynomials $L_n^{(\alpha)}(x)$, Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, Rice polynomials $H_n^{(\alpha,\beta)}(\rho,\mu,x)$ and Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ defined by (1.3.7), (1.5.1), (1.5.6), (1.5.11), (1.5.23), and (1.5.27) respectively, are also obtained.

Many known results of Chatterjea [15], Feldheim [27], Khan and Shukla [36], Pathan and Bin Saad [67], Srivastava [84, 86] and Srivastava and Manocha [94] are shown as special cases of these generating relations.

3.2 DOUBLE GENERATING RELATIONS

Consider the generating functions for the confluent hypergeometric function ${}_1F_1$ {[26], p.403 (2.2) and [25], p.12 (6.12)}

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\; x^m\; s^n}{((g))_{m+n}\; m!\, n!}\; {}_1F_1\left[\begin{matrix}-m;\\ p\;;\end{matrix} -y\right] {}_1F_1\left[\begin{matrix}-n;\\ p'\;;\end{matrix} -t\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\,(xy)^m\,(st)^n}{((g))_{m+n}\,(p)_m\,(p')_n\; m!\, n!}$$

$$F^{D:0;\,0}_{G:0;0}\left[\begin{matrix}(d)+m+n:-;-;\\ (g)+m+n:-;-;\end{matrix}\; x,s\right] \qquad \text{...... (3.2.1)}$$

and

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\,(a)_n\,(1-a-m-n)_m\, x^{m+n}}{((g))_{m+n}\; m!\, n!}\; {}_1F_1\left[\begin{matrix}-m\;;\\ 1-a-m-n;\end{matrix}\; y\right]$$

$$= {}_DF_G\left[\begin{matrix}(d);\\ (g);\end{matrix}\; xy\right]. \qquad \text{...... (3.2.2)}$$

In (3.2.1) if we replace $-y$, $-t$ by z, y respectively and we put $s = -x$, we obtain

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\; x^m\; (-x)^n}{((g))_{m+n}\; m!\, n!}\; {}_1F_1\left[\begin{matrix}-m;\\ p\;;\end{matrix} z\right] {}_1F_1\left[\begin{matrix}-n;\\ p'\;;\end{matrix} y\right]$$

$$= F\begin{matrix} D:0;\,0 \\ G:1;1 \end{matrix}\left[\begin{matrix} (d):-;-; \\ (g):p;p'; \end{matrix} -xz,xy\right]. \qquad \text{...... (3.2.3)}$$

Now, if in (3.2.3), we replace y by yt, multiply both sides by $t^{l'-1}\ e^{-st}$ and take Laplace transform and use the result {[94], p.219 (6)}.

$$L\left\{t^{\lambda-1}\ {}_AF_B\left[\begin{matrix} a_1,....,a_A; \\ b_1,....,b_B; \end{matrix} zt\right]:S\right\} = \frac{\Gamma(\lambda)}{S^{\lambda}}\ {}_{A+1}F_B\left[\begin{matrix} \lambda, a_1,......,a_A;\ \frac{z}{s} \\ b_1,........,b_B;\ s \end{matrix}\right] \qquad \text{...... (3.2.4)}$$

$\mathrm{Re}(\lambda)>0, A\leq B; \mathrm{Re}(s)>0\ \ if\ \ A<B; \mathrm{Re}(s)>\mathrm{Re}(z)\ \ if\ \ A=B$

we obtain

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\ x^m\ (-x)^n}{((g))_{m+n}\ \ m!\ n!}\ {}_1F_1\left[\begin{matrix} -m; \\ p\ ; \end{matrix} z\right] {}_2F_1\left[\begin{matrix} -n\ ,\ l'; \\ p'\ ; \end{matrix} y\right]$$

$$= F\begin{matrix} D:0;\,1 \\ G:1;1 \end{matrix}\left[\begin{matrix} (d):-;l'; \\ (g):p;p'; \end{matrix} -xz,xy\right]. \qquad \text{...... (3.2.5)}$$

Next, if in (3.2.5), we replace z by zt, multiply both sides by $t^{l-1}\ e^{-st}$ and take Laplace transform and use (3.2.4), we obtain

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\ x^m\ (-x)^n}{((g))_{m+n}\ \ m!\ n!}\ {}_2F_1\left[\begin{matrix} -m\ ,\ l; \\ p\ ; \end{matrix} z\right] {}_2F_1\left[\begin{matrix} -n\ ,\ l'; \\ p'\ ; \end{matrix} y\right]$$

$$= F\begin{matrix} D:1;1 \\ G:1;1 \end{matrix}\left[\begin{matrix} (d)\ :l;\ l'; \\ (g):\ p;\ p'; \end{matrix} -xz,\ xy\right]. \qquad \text{...... (3.2.6)}$$

Now, if the Gauss hypergeometric function ${}_2F_1(\cdot)$ and the confluent hypergeometric function ${}_1F_1(\cdot)$ on the left of (3.2.5) and (3.2.6) are replaced by their representation as Laguerre, Jacobi and Legendre polynomails with the help of the definitions {[71], p.200 (1)}, {[83], p.593 (20)} and{[25], p.8 (5.6)}.

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\ {}_1F_1\left[\begin{matrix} -n\ ; \\ \alpha+1; \end{matrix} x\right], \qquad \ldots\ldots (3.2.7)$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix} -n, 1+\alpha+\beta+n\ ; \\ 1+\alpha\ ; \end{matrix} \frac{1-x}{2}\right] \qquad \ldots\ldots (3.2.8)$$

and

$$P_m(x) = \left[x+(x^2-1)^{1/2}\right]^m\ {}_2F_1\left[\begin{matrix} -m, 1/2; \\ 1\ ; \end{matrix} \frac{2(x^2-1)^{1/2}}{x+(x^2-1)^{1/2}}\right], \qquad \ldots\ldots (3.2.9)$$

following results are obtained:

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\ x^m\ (-x)^n}{((g))_{m+n}\ (\alpha+1)_m\ n!}\ L_m^{(\alpha)}(z)\ {}_2F_1\left[\begin{matrix} -n\ ,\ l'; \\ p'\ ; \end{matrix} y\right]$$

$$= F_{G:1;1}^{D:0;1}\left[\begin{matrix} (d)\ :-;\ l'\ ; \\ (g):\alpha+1;\ p'\ ; \end{matrix} -xz,\ xy\right], \qquad \ldots\ldots (3.2.10)$$

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\ x^m\ (-x)^n}{((g))_{m+n}\ (\alpha+1)_m\ n!}\ P_m^{(\alpha,\beta-m)}(z)\ {}_2F_1\left[\begin{matrix} -n\ ,\ l'; \\ p'\ ; \end{matrix} y\right]$$

$$= F_{G:1;1}^{D:1;1}\left[\begin{matrix} (d)\ :\alpha+\beta+1;\ l';\ \\ (g):\alpha+1;\ p'\ ; \end{matrix} \frac{-x(1-z)}{2},\ xy\right], \qquad \ldots\ldots (3.2.11)$$

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\ x^m}{((g))_{m+n}\ m!\ n!}\left[\frac{-x}{y+(y^2-1)^{1/2}}\right]^n\ {}_2F_1\left[\begin{matrix} -m\ ,\ l; \\ p\ ; \end{matrix} z\right] P_n(y)$$

$$= F_{G:1;1}^{D:1;1}\left[\begin{matrix} (d)\ :l;\ 1/2; \\ (g):p;\ 1\ ; \end{matrix} -xz, \frac{2x(y^2-1)^{1/2}}{y+(y^2-1)^{1/2}}\right], \qquad \ldots\ldots (3.2.12)$$

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\ x^m\ (-x)^n}{((g))_{m+n}\ (\alpha+1)_m\ (\gamma+1)_n}\ P_m^{(\alpha,\beta-m)}(z)\ P_n^{(\gamma,\delta-n)}(y)$$

$$= F\begin{matrix} D:1;1 \\ G:1;1 \end{matrix}\left[\begin{matrix} (d): \alpha+\beta+1;\ \gamma+\delta+1; & \frac{-x(1-z)}{2}, \frac{x(1-y)}{2} \\ (g): \quad \alpha+1; \gamma+1 \quad ; & \end{matrix}\right] \qquad \ldots. (3.2.13)$$

and

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}}{((g))_{m+n}\ m!\ n!} \left[\frac{x}{z+(z^2-1)^{1/2}}\right]^m \left[\frac{-x}{y+(y^2-1)^{1/2}}\right]^n P_m(z)\ P_n(y)$$

$$= F\begin{matrix} D:1;1 \\ G:1;1 \end{matrix}\left[\begin{matrix} (d): 1/2;\ 1/2; & \\ (g): \quad 1;\ 1 \quad ; & \end{matrix} \xi_1, \xi_2\right]. \qquad \ldots\ldots (3.2.14)$$

where

$$\xi_1 = \frac{-2x(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}, \ \xi_2 = \frac{2x(y^2-1)^{1/2}}{y+(y^2-1)^{1/2}}.$$

3.3 SPECIAL CASES

Now we mention some interesting special cases of the equations (3.2.10) – (3.2.14).

On taking D =1, G =0, d = λ and $l'=p'$ in (3.2.10), we get

$$[1-x(y-1)]^{-\lambda} \sum_{m=0}^{\infty} \frac{(\lambda)_m}{(1+\alpha)_m} \left[\frac{x}{1-x(y-1)}\right]^m L_m^{(\alpha)}(z)$$

$$=(1-xy)^{-\lambda}\ {}_1F_1\left[\begin{matrix} \lambda & ; \frac{xz}{xy-1} \\ 1+\alpha ; & \end{matrix}\right] \qquad \ldots\ldots (3.3.1)$$

which for y = 1 reduces to a known result {[94], p. 132 (5)}.

On setting $l'=p'$ in (3.2.11), we get

$$\sum_{m,n=0}^{\infty} \frac{((d))_m \; x^m}{((g))_m \; (\alpha+1)_m} P_m^{(\alpha,\beta-m)}(z)\,{}_{D+m}F_{G+m}\left[\begin{matrix}(d)+m \; ; \\ (g)+m \; ;\end{matrix} x(y-1)\right]$$

$$= F\begin{matrix}D:1;0\\G:1;0\end{matrix}\left[\begin{matrix}(d):\alpha+\beta+1; \; -; \\ (g): \quad \alpha+1; \; - \;\; ;\end{matrix} \frac{-x(1-z)}{2}, xy\right] \qquad \text{...... (3.3.2)}$$

which for y=1, reduces to the following interesting generating function for Jacobi polynomials given by

$$\sum_{m,n=0}^{\infty} \frac{((d))_m \; x^m}{((g))_m \; (\alpha+1)_m} P_m^{(\alpha,\beta-m)}(z)$$

$$= F\begin{matrix}D:1;0\\G:1;0\end{matrix}\left[\begin{matrix}(d):\alpha+\beta+1; \; -; \\ (g): \quad \alpha+1; \; - \;\; ;\end{matrix} \frac{-x(1-z)}{2}, x\right] \qquad \text{...... (3.3.3)}$$

On putting D=2, G=1, $d_1=\lambda$, $d_2=\rho$ and $g_1=\mu$ in (3.3.3) and using Euler's transformation {[94], p. 33 (19)}, we get

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m (\rho)_m \; x^m}{(\mu)_m \; (\alpha+1)_m} P_m^{(\alpha,\beta-m)}(z)$$

$$=(1-x)^{-\lambda} \; F\begin{matrix}1:1;2\\1:0;1\end{matrix}\left[\begin{matrix}\lambda:\mu-\rho;\rho,1+\alpha+\beta \; ; \\ \mu: \qquad -;\alpha+1 \qquad ;\end{matrix} \frac{x}{x-1} \; \frac{x(1-z)}{2(x-1)}\right] \qquad \text{...... (3.3.4)}$$

which is a known result of Srivastava {[84], p.65}

For D = G = 0 and $l'=p'$ (3.2.11) reduces to

$$\sum_{m=0}^{\infty} \frac{x^m}{(1+\alpha)_m} P_m^{(\alpha,\beta-m)}(z) = e^x \; {}_1F_1\left[\begin{matrix}1+\alpha+\beta; \\ \alpha+1 \;\; ;\end{matrix} \frac{-x(1-z)}{2}\right], \qquad \text{...... (3.3.5)}$$

which, in view of Kummer's first theorem {cf. [94], p.37 (7)}

$${}_1F_1\left[\begin{matrix}a; \\ c;\end{matrix} z\right] = e^x \; {}_1F_1\left[\begin{matrix}c-a; \\ c \;\; ;\end{matrix} -z\right] \qquad \text{...... (3.3.6)}$$

yields Feldheim's formula {[27], p.120 (12)}.

$$\sum_{m=0}^{\infty} \frac{x^m}{(1+\alpha)_m} P_n^{(\alpha,\beta-m)}(z) = e^x \, {}_1F_1\left[\begin{matrix} -\beta \, ; \\ \alpha+1; \end{matrix} \frac{1}{2}(1-z)x\right]. \qquad \ldots. (3.3.7)$$

Now, in (3.2.12) let $D = 1$, $G = 0$, $d = p = \lambda$ and setting $y = 1$, we get

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} \left(\frac{x}{1+x}\right)^m {}_2F_1\left[\begin{matrix} -m, l; \\ \lambda \;\; ; \end{matrix} z\right] = (1+x)^{\lambda} (1+xz)^{-l}, \qquad \ldots. (3.3.8)$$

which, on replacing $\frac{x}{1+x}$ by t reduces to a known result {[94], p.293 (12)}.

On taking $D=1, G=0, d=\alpha+1, \alpha=\gamma, \beta=\delta$ in (3.2.13), we get

$$\sum_{m,n=0}^{\infty} \frac{(1+\alpha)_{m+n} \, x^m \, (-x)^n}{(\alpha+1)_m \, (\alpha+1)_n} P_m^{(\alpha,\beta-m)}(z) \, P_n^{(\alpha,\beta-n)}(y)$$

$$= F_2\left[\alpha+1, \alpha+\beta+1, \alpha+\beta+1; \alpha+1, \alpha+1; \frac{-x(1-z)}{2}, \frac{x(1-y)}{2}\right] \qquad \ldots. (3.3.9)$$

on using the result {[93], p.305 (108)}.

$$F_2\left[\alpha, \beta, \beta'; \alpha, \alpha; x, y\right]$$

$$= (1-x)^{-\beta} (1-y)^{-\beta'} \, {}_2F_1\left[\beta, \beta'; \alpha; \frac{xy}{(1-x)(1-y)}\right], \qquad \ldots. (3.3.10)$$

(3.3.9) reduces to a know result due to Pathan and Bin Saad {[67], p.149 (2.16)}

$$\sum_{m,n=0}^{\infty} \frac{(1+\alpha)_{m+n} \, x^m \, (-x)^n}{(\alpha+1)_m \, (\alpha+1)_n} P_m^{(\alpha,\beta-m)}(z) \, P_n^{(\alpha,\beta-n)}(y)$$

$$=\left[\left(1-\frac{x}{2}+\frac{xy}{2}\right)\left(1+\frac{x}{2}-\frac{xy}{2}\right)\right]^{-(\alpha+\beta+1)}$$

$${}_2F_1\left[\begin{matrix}\alpha+\beta+1,\alpha+\beta+1; & \dfrac{-x^2(1-z)(1-y)}{(2+x-xz)(2-x+xz)} \\ \alpha+1 & ; \end{matrix}\right]. \qquad \text{......} (3.3.11)$$

Buschman and Srivastava [11] gave a number of reducible cases of the Kampé de Fériet function including the formula

$$F^{A:1;1}_{C:1;1}\left[\begin{matrix}(a):b;b; \\ (c):d;d;\end{matrix} x,-x\right]$$

$$= {}_{2A+2}F_{2C+3}\left[\begin{matrix}(a/2),(a/2+1/2),b, & d-b & ; \\ (c/2),(c/2+1/2),d,d/2,d/2+1/2; & \end{matrix} 4^{A-C-1}x^2\right] \qquad \text{......} (3.3.12)$$

Now, if we put $z = y$, $\alpha = \gamma$ and $\beta = \delta$, the right hand side of (3.2.13) can be reduce to a single hypergeometric functions by means of (3.3.12). Hence,

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\, x^m\, (-x)^n}{((g))_{m+n}\, (\alpha+1)_m\, (\alpha+1)_n}\, P_m^{(\alpha,\beta-m)}(y)\, P_n^{(\alpha,\beta-n)}(y)$$

$$= {}_{2D+2}F_{2G+3}\left[\begin{matrix}(d/2),(d/2+1/2),\alpha+\beta+1, & -\beta & ; \\ (g/2),(g/2+1/2), \quad \alpha+1 & ,(\alpha+1)/2,(\alpha+1)/2+1/2; \end{matrix}\right.$$

$$\left. 4^{D-G-1}\left[\frac{x(1-y)}{2}\right]^2\right], \qquad \text{......} (3.3.13)$$

which for $D = 1$, $G = 0$, $d_1 = \alpha + 1$, reduces to

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\ x^m\ (-x)^n}{(\alpha+1)_m\ (\alpha+1)_n}\ P_m^{(\alpha,\beta-m)}(y)\ P_n^{(\alpha,\beta-n)}(y)$$

$$= {}_2F_1\left[\begin{matrix}1+\alpha+\beta,-\beta; \\ \alpha+1\quad ;\end{matrix}\left(\frac{x(1-y)}{2}\right)^2\right]. \qquad \ldots\ldots (3.3.14)$$

Similarly in case of equation (3.2.14), if we put $z = y$ and use (3.3.12), we get

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}}{((g))_{m+n}\ m!\ n!}\left[\frac{x}{y+(y^2-1)^{1/2}}\right]^m \left[\frac{-x}{y+(y^2-1)^{1/2}}\right]^n P_m(y)\ P_n(y)$$

$$= {}_{2D+1}F_{2G+2}\left[\begin{matrix}(d/2),(d/2+1/2),\ 1/2\ ; \\ (g/2),(g/2+1/2),\ 1\ ,1\ ;\end{matrix} 4^{D-G-1}\ \xi^2\right] \qquad \ldots\ldots (3.3.15)$$

where $\xi = \dfrac{2x(y^2-1)^{1/2}}{y+(y^2-1)^{1/2}}$

which for $D = 1,\ G = 0,\ d_1 = 1$, reduces to

$$\sum_{m,n=0}^{\infty} \frac{(m+n)!}{m!\ n!}\left[\frac{x}{y+(y^2-1)^{1/2}}\right]^m \left[\frac{-x}{y+(y^2-1)^{1/2}}\right]^n P_m(y)\ P_n(y)$$

$$= {}_2F_1\left[\begin{matrix}1/2,1/2; \\ 1\quad ;\end{matrix}\xi^2\right]. \qquad \ldots\ldots (3.3.16)$$

3.4 FURTHER DOUBLE GENERATING RELATIONS

Now in (3.2.6), by repeated application of inverse Laplace and Laplace transform with the help of the result {[94], p.219 (7)}.

$$L^{-1}\left\{s^{-\lambda}\ {}_AF_B\left[\begin{matrix}a_1,\ldots,a_A; \\ b_1,\ldots,b_B;\end{matrix}\ \frac{z}{s}\right]:t\right\} = \frac{t^{\lambda-1}}{\Gamma(\lambda)}\ {}_AF_{B+1}\left[\begin{matrix}a_1,\ldots\ldots,a_A\ ; \\ \lambda,b_1,\ldots\ldots,b_B;\end{matrix}\ zt\right]; \qquad \ldots\ldots (3.4.1)$$

$\mathrm{Re}(\lambda) > 0,\ A \le B+1$

we shall arrive to the following result:

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\; x^{m}\; (-x)^{n}}{((g))_{m+n}\;\; m!\; n!}\; {}_{1+L}F_{K+1}\left[\begin{matrix} -m,(l); \\ p,(k); \end{matrix}\; z\right] {}_{1+R}F_{S+1}\left[\begin{matrix} -n,(r); \\ p',(s)\;; \end{matrix}\; y\right]$$

$$= F^{D:\;\; L;R}_{G:K+1;S+1}\left[\begin{matrix} (d)\; :(l);\; (r) \quad ; \\ (g):\, p,(k);\; p',\,(s); \end{matrix}\; -xz,\, xy\right]. \qquad \text{...... (3.4.2)}$$

Similarly, in case of (3.2.2), if we adopt the same analysis that is employed to obtain (3.4.2), we get

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\; (a)_{n}\, (1-a-m-n)_{m}\; x^{m+n}}{((g))_{m+n}\;\; m!\; n!}\; {}_{1+L}F_{K+1}\left[\begin{matrix} -m\;,\,(l) \quad ; \\ 1-a-m-n,(k); \end{matrix}\, y\right]$$

$$= {}_{D+L}F_{G+K}\left[\begin{matrix} (d),\,(l); \\ (g),(k); \end{matrix}\; xy\right]. \qquad \text{...... (3.4.3)}$$

Further, in view of the following definitions of Laguerre polynomials of two variables $L_{n}^{(\alpha,\beta)}(x,x)$ {[15], p.266 (12)} and generalized Rice polynomials $H_{n}^{(\alpha,\beta)}(\nu,\sigma,x)$ {cf. [94], p.140 (13)}.

$$L_{n}^{(\alpha,\beta)}(x,x)=\frac{(1+\alpha)_{n}\,(1+\beta)_{n}}{(n!)^{2}}\; {}_{3}F_{3}\left[\begin{matrix} -n,\dfrac{\alpha+\beta+1}{2},\dfrac{\alpha+\beta+2}{2}\;; \\ \alpha+1,\; \beta+1,\alpha+\beta+1\;; \end{matrix}\; 4x\right] \qquad \text{...... (3.4.4)}$$

and

$$H_{n}^{(\alpha,\beta)}(\nu,\sigma,x)=\binom{\alpha+n}{n}\; {}_{3}F_{2}\left[\begin{matrix} -n,\alpha+\beta+n+1,\nu; \\ \alpha+1,\sigma \quad ; \end{matrix}\; x\right], \qquad \text{...... (3.4.5)}$$

equation (3.4.2) would give us the following interesting generating relations:

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\, x^{m}\, (-x)^{n}\, m!\, n!}{((g))_{m+n} (1+\alpha)_m\, (1+\beta)_m\, (1+\gamma)_n\, (1+\delta)_n} L_m^{(\alpha,\beta)}(z,z)\, L_n^{(\gamma,\delta)}(y,y)$$

$$= F_{G:3;3}^{D:2;2}\left[\begin{matrix}(d): \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}; \frac{\gamma+\delta+1}{2}, \frac{\gamma+\delta+2}{2}; \\ (g): \alpha+1, \beta+1, \alpha+\beta+1; \gamma+1, \delta+1, \gamma+\delta+1;\end{matrix} -4xz, 4xy\right] \quad \ldots\ldots (3.4.6)$$

and

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\, x^{m}\, (-x)^{n}}{((g))_{m+n} (1+\alpha)_m\, (1+\gamma)_n} H_m^{(\alpha,\beta-m)}(\mu,\rho,z)\, H_n^{(\gamma,\delta-n)}(\nu,\sigma,y)$$

$$= F_{G:2;2}^{D:2;2}\left[\begin{matrix}(d): \alpha+\beta+1, \mu; \gamma+\delta+1, \nu; \\ (g): \quad \alpha+1, \quad \rho; \quad \gamma+1, \quad \sigma;\end{matrix} -xz, xy\right]. \quad \ldots\ldots (3.4.7)$$

Yet, another generating relation of interest for Appell's hypergeometric function F_4 defined by (1.3.7) is as follows:

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\, x^{m}\, (-x)^{n}}{((g))_{m+n}\, m!\, n!} F_4[-m, \alpha; \gamma, \gamma'; z, z]\, F_4[-n, \beta; \delta, \delta'; y, y]$$

$$= F_{G:3;3}^{D:3;3}\left[\begin{matrix}(d): \alpha, \tfrac{1}{2}(\gamma+\gamma'), \tfrac{1}{2}(\gamma+\gamma'-1); \beta, \tfrac{1}{2}(\delta+\delta'), \tfrac{1}{2}(\delta+\delta'-1); \\ (g): \quad \gamma, \gamma' \quad , \quad \gamma+\gamma'-1 \quad ; \quad \delta, \delta' \quad , \quad \delta+\delta'-1 \quad ;\end{matrix} -4xz, 4xy\right].$$

$$\ldots\ldots (3.4.8)$$

Now, on multiplying both sides of (3.4.3) by

$$t^{u_1-1}\, e^{-st}\, F_{V:F';\cdots\cdots;F^{(n)}}^{U:E';\cdots\cdots;E^{(n)}}[y_1 t, \cdots\cdots, y_n t]$$

where $F_{V:F';\cdots\cdots;F^{(n)}}^{U:E';\cdots\cdots;E^{(n)}}[x_1, \cdots\cdots, x_n]$ is the generalized Kampé de Fériet Function

of several variables defined by (1.4.6)

$$F_{C:D';\cdots\cdots;D^{(n)}}^{A:B';\cdots\cdots;B^{(n)}}[x_1, \cdots\cdots, x_n]$$

$$= F\begin{matrix} A:B';\cdots\cdots;B^{(n)} \\ C:D';\cdots\cdots;D^{(n)} \end{matrix}\left[\begin{matrix}(a):(b');\cdots\cdots;(b^{(n)})\,; \\ (c):(d');\cdots\cdots;(d^{(n)})\,; \end{matrix} x_1,\cdots\cdots,x_n\right]$$

$$= \sum_{m_1,\cdots\cdots,m_n=0}^{\infty} \frac{((a))_{m_1+\cdots\cdots+m_n}\,((b'))_{m_1}\cdots\cdots((b^{(n)}))_{m_n}\, x_1^{m_1}\cdots\cdots x_n^{m_n}}{((c))_{m_1+\cdots\cdots+m_n}\,((d'))_{m_1}\cdots\cdots((d^{(n)}))_{m_n}\, m_1!\cdots\cdots m_n!} \quad \cdots\cdots (3.4.9)$$

and replacing y by yt and taking Laplace transform with the help of (3.2.4), we get

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\,(a)_n\,(1-a-m-n)_m\, x^{m+n}}{((g))_{m+n}\; m!\, n!} \sum_{m_1,\cdots\cdots,m_n=0}^{\infty} \Phi(m_1,\cdots\cdots,m_n)$$

$${}_{2+L}F_{K+1}\left[\begin{matrix} u_1+m_1+\cdots\cdots+m_n,-m,(l); \\ 1-a-m-n,(k)\quad ; \end{matrix} y\right]$$

$$= \sum_{m_1,\cdots\cdots,m_n=0}^{\infty} \Phi(m_1,\cdots\cdots,m_n)\,{}_{1+D+L}F_{G+K}\left[\begin{matrix} u_1+m_1+\cdots\cdots+m_n,(d),(l); \\ (g),(k)\quad ; \end{matrix} xy\right] \cdots (3.4.10)$$

where $\Phi(m_1,\cdots\cdots,m_n) = \dfrac{((u))_{m_1+\cdots\cdots+m_n}\,((e'))_{m_1}\cdots\cdots((e^{(n)}))_{m_n}\, y_1^{m_1}\cdots\cdots y_n^{m_n}}{((v))_{m_1+\cdots\cdots+m_n}\,((f'))_{m_1}\cdots\cdots((f^{(n)}))_{m_n}\, m_1!\cdots\cdots m_n!}$.

Now, in (3.4.10) by repeated application of inverse Laplace and Laplace transform with the help of the results (3.2.4) and (3.4.1), we obtain the following interesting result after a little simplification:

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\,(a)_n\,(1-a-m-n)_m\, x^{m+n}}{((g))_{m+n}\; m!\, n!}$$

$$\times F\begin{matrix} U:E';\cdots;E^{(n)}\;;L+1 \\ V:F';\cdots;F^{(n)}\;;K+1 \end{matrix}\left[\begin{matrix}(u):(e');\cdots;(e^{(n)})\;; & (l),-m & ; \\ (v):(f');\cdots;(f^{(n)})\;; & (k),1-a-m-n; \end{matrix} y_1,\cdots,y_n,y\right]$$

$$= F\begin{matrix} U:E';\cdots\cdots;E^{(n)}\;;D+L \\ V:F';\cdots\cdots;F^{(n)}\;;G+K \end{matrix}\left[\begin{matrix}(u):(e');\cdots\cdots;(e^{(n)})\;;(d),(l); \\ (v):(f');\cdots\cdots;(f^{(n)})\;;(g),(k); \end{matrix} y_1,\cdots\cdots,y_n,\, xy\right] \ldots (3.4.11)$$

3.5 SPECIAL CASES

On putting $L = K = 2$, $R = 1$, $S = 0$ in (3.4.2) and using (3.4.4), we get

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\, x^m\, (-x)^n\, m!}{((g))_{m+n}\, (\alpha+1)_m\, (\beta+1)_m\, n!}\, L_m^{(\alpha,\beta)}(z,z)\; {}_2F_1\left[\begin{matrix} -n,\, r\,; \\ p'\;; \end{matrix}\; y\right]$$

$$= F\begin{matrix} D:2;1 \\ G:3;1 \end{matrix}\left[\begin{matrix} (d):\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+1}{2};r; \\ (g):\alpha+\beta+1,\alpha+1,\beta+1;p'; \end{matrix}\; -4xz, xy\right]. \qquad \ldots\ldots (3.5.1)$$

Now, if we take D = G = 0, r = p' in (3.5.1) and use the formula {[93], p.315 (151)}

$${}_0F_1\left[\begin{matrix} -\,; \\ a\,; \end{matrix}\; x\right]\; {}_0F_1\left[\begin{matrix} -\,; \\ b\,; \end{matrix}\; x\right] = {}_2F_3\left[\begin{matrix} \frac{a+b}{2},\frac{a+b-1}{2}\,; \\ a,b,a+b-1\,; \end{matrix}\; 4x\right], \qquad \ldots\ldots (3.5.2)$$

we get

$$\sum_{m=0}^{\infty} \frac{m!\, L_m^{(\alpha,\beta)}(z,z)\, x^m}{(\alpha+1)_m\, (\beta+1)_m}$$

$$= \exp(x)\; {}_0F_1\left[\begin{matrix} -\,; \\ \alpha+1; \end{matrix}\; -xz\right]\; {}_0F_1\left[\begin{matrix} -\,; \\ \beta+1; \end{matrix}\; -xz\right] \qquad \ldots\ldots (3.5.3)$$

which is a known result {[15], p.263 (2)}.

If in (3.5.1) we let $D = 1$, $G = 0$, $d = c$ and put $r = p'$, we get

$$\left[1 - x(y-1)^{-c}\right] \sum_{m=0}^{\infty} \frac{m!\, (c)_m\, L_m^{(\alpha,\beta)}(z,z)\, x^m}{(\alpha+1)_m\, (\beta+1)_m}$$

$$= (1-xy)^{-c}\; {}_3F_3\left[\begin{matrix} c,\frac{\alpha+\beta+2}{2},\frac{\alpha+\beta+1}{2}\,; \\ \alpha+1,\, \beta+1,\alpha+\beta+1\,; \end{matrix}\; \frac{4xz}{xy-1}\right]. \qquad \ldots\ldots (3.5.4)$$

On putting y = 1 in (3.5.4) and using the formula {[93], p.315 (150)}.

$$\Psi_2\,(a;b,c;x,x)={}_3F_3\left[\begin{matrix}a,\dfrac{b+c}{2},\dfrac{b+c-1}{2}; \\ b,c,b+c-1\,;\end{matrix}\;4x\right], \qquad \text{......} (3.5.5)$$

we get

$$\sum_{m=0}^{\infty}\frac{m!(c)_m\,L_m^{(\alpha,\beta)}(z,z)\,x^m}{(\alpha+1)_m\,(\beta+1)_m}$$

$$=(1-x)^{-c}\,\Psi_2\left(c;\alpha+1,\beta+1;\frac{xz}{x-1},\frac{xz}{x-1}\right) \qquad \text{......} (3.5.6)$$

which is a known result {[36], p.157 (2.5)}.

Now, on taking $L = 2, K = R = 1, S = 0$ in (3.4.2) and using (3.4.5), we get

$$\sum_{m,n=0}^{\infty}\frac{((d))_{m+n}\,x^m\,(-x)^n}{((g))_{m+n}(1+\alpha)_m\;n!}\;H_m^{(\alpha,\beta-m)}(\mu,\rho,z)\;{}_2F_1\left[\begin{matrix}-n,r; \\ p'\;;\end{matrix}\;y\right]$$

$$=F\begin{matrix}D:2;1\\G:2;1\end{matrix}\left[\begin{matrix}(d):\alpha+\beta+1,\mu;\;r\;; \\ (g):\;\;\alpha+1,\;\;\;\rho;\;p';\end{matrix}\;-xz,xy\right]. \qquad \text{......} (3.5.7)$$

If in (3.5.7), we put $r=p'$, we obtain

$$\sum_{m=0}^{\infty}\frac{((d))_m\,x^m}{((g))_m(1+\alpha)_m}\;H_m^{(\alpha,\beta-m)}(\mu,\rho,z)\;{}_{D+m}F_{G+m}\left[\begin{matrix}(d)+m; \\ (g)+m;\end{matrix}\;x(y-1)\right]$$

$$=F\begin{matrix}D:2;0\\G:2;0\end{matrix}\left[\begin{matrix}(d):\alpha+\beta+1,\mu;\;-; \\ (g):\;\;\alpha+1,\;\;\;\rho;\;-;\end{matrix}\;-xz,xy\right]. \qquad \text{......} (3.5.8)$$

which for y = 1 reduces to a known result due to Srivastava {[86], p.77 (3.3)}.

$$\sum_{m=0}^{\infty} \frac{((d))_m x^m}{((g))_m (1+\alpha)_m} \; H_m^{(\alpha,\beta-m)}(\mu,\rho,z)$$

$$= F\begin{matrix} D:2;0 \\ G:2;0 \end{matrix}\left[\begin{matrix} (d):\alpha+\beta+1,\mu;\,-; \\ (g):\;\alpha+1,\quad \rho;\,-; \end{matrix} -xz, x\right]. \qquad \text{...... (3.5.9)}$$

where $((d))_m = \prod_{j=1}^{D} (d_j)_m$, with similar interpretation for $((g))_m$

Finally, in terms of Appell's function F_2, Kampé de Fériet function of two variables $F\begin{matrix} A:B;B' \\ C:D;D' \end{matrix}[x,y]$ and Lauricella's function $F_A^{(n)}$, defined by (1.3.5), (1.3.12) and (1.4.1) respectively, equation (3.4.11) gives the following interesting special cases:

$$\sum_{m,n=0}^{\infty} \frac{(a)_n (1-a-m-n)_m x^{m+n}}{m!\, n!} \; F_2\,[u,e,-m;f,1-a-m-n;y',y]$$

$$= (1-xy)^{-u}\; {}_2F_1\left[u,e;f;\frac{y'}{1-xy}\right], \qquad \text{...... (3.5.10)}$$

$$\sum_{m,n=0}^{\infty} \frac{(a)_n (1-a-m-n)_m x^{m+n}}{m!\, n!}$$

$$F\begin{matrix} U:E;L+1 \\ V:F;K+1 \end{matrix}\left[\begin{matrix} (u):(e); & -m,(l) & ; \\ (v):(f); & 1-a-m-n,(k) & ; \end{matrix} y', y\right]$$

$$= F\begin{matrix} U:E;L \\ V:F;K \end{matrix}\left[\begin{matrix} (u):(e);(l) & ; \\ (v):(f);(k) & ; \end{matrix} y', xy\right] \qquad \text{...... (3.5.11)}$$

and

$$\sum_{m,n=0}^{\infty} \frac{(d)_{m+n} (a)_n (1-a-m-n)_m x^{m+n}}{(g)_{m+n}\; m!\, n!}$$

$$F_A^{(n+1)}\;[u,e_1,\cdots\cdots,e_n,-m;f_1,\cdots\cdots,f_n,1-a-m-n;y_1,\cdots,y_n,y]$$

$$= F_A^{(n+1)}\;[u,e_1,\cdots\cdots,e_n,d;f_1,\cdots\cdots,f_n,g;y_1,\cdots,y_n,xy]. \qquad \text{...... (3.5.12)}$$

CHAPTER-IV

Fractional Derivatives and Generating Functions Involving Hypergeometric Functions of three variables

CHAPTER-IV

FRACTIONAL DERIVATIVES AND GENERATING FUNCTIONS INVOLVING HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

4.1 INTRODUCTION

The theory of fractional calculus is concerned with nth derivative and n-fold integral when it becomes an arbitrary parameter.

Starting with $y = x^n$, n a positive integer, S.F. Lacroix [40] gave the mth derivative to be

$$\frac{d^m y}{d x^m} = \frac{n!}{(n-m)!} x^{n-m}. \qquad .....(4.1.1)$$

With the usual application of the Gamma function and by replacing m by ½ and n by any positive real number a, he obtained the formula

$$\frac{d^{1/2} y}{d x^{1/2}} = \frac{\Gamma(a+1)}{\Gamma(a+\frac{1}{2})} x^{a-\frac{1}{2}}. \qquad .....(4.1.2)$$

He gave the example for y = x and derived.

$$\frac{d^{1/2}}{d x^{1/2}} (x) = \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} = \frac{2\sqrt{x}}{\sqrt{\pi}}. \qquad .....(4.1.3)$$

This result is the same yielded by the present day Rieman-Liouvill definition of a fractional calculus.

Liouvill defines the fractional derivative of order ν by

$$D_x^v f(x) = \sum_{n=0}^{\infty} c_n \; a_n^v \; e^{a_n x} \qquad .....(4.1.4)$$

where

$$f(x) = \sum_{n=0}^{\infty} c_n \; e^{a_n x} \qquad .....(4.1.5)$$

The formula (4.1.4) is known as Lioville's [43] first definition.

Lioville's second method was applied to explicit function of the form x^{-a}, $a > 0$. He consider the integral

$$I = \int_0^{\infty} u^{a-1} e^{-xu} \, du \qquad .....(4.1.6)$$

The transformation $xu = t$, gives the result

$$x^{-a} = \frac{1}{\Gamma(a)} I \qquad .....(4.1.7)$$

then with the use of (4.1.1) he obtained the following result

$$D_x^v \; x^{-a} = \frac{(-1)^v \; \Gamma(a+v)}{\Gamma(a)} \; x^{-a} - v. \qquad .....(4.1.8)$$

Liouville was successful in applying these definitions to problems in potential theory. The first definition is restricted to certain values of ν and the second method is not suitable to a wide class of functions.

A more detailed exposition could be found in the historical remarks of Ross [73-74] and Mikolas [49] as well as in the surveys of Sneden [78-79], Al-Bassam [2] and Samko [75].

Recently the Fractional Calculus has been investigated by many mathematician. In particular, K. Nishimoto [51-65] has presented a systematic account of the theory and applications of fractional calculus in a number of areas (such as ordinary and partial differential equation, special functions and summation of series).

In this chapter we apply the concept of Nishimoto's fractional calculus (N-fractional calculus) to prove some fractional derivatives formula involving hypergeometric functions of three variables, to be used in our investigation to obtain some linear, bilinear and bilateral generating functions involving triple hypergeometric functions $F^{(3)}$ $[x, y, z]$ defined by (1.4.11) and Luricella-Saran's functions of three variables $F_G, F_N, F_S,$ defined by {(1.4.7), (1.4.9), (1.4.10)}.

4.2 THE FRACTIONAL DERIVATIVES FORMULA

In what follows, we prove the following fractional derivatives formula:

$$D_x^{\mu}\left[x^{\alpha}(x-1)^{\beta}\left(1-\frac{w_1}{x-1}\right)^{-\gamma}\left(1-\frac{w_2}{x-1}\right)^{-\delta}\right]$$

$$= A.F_G\left[-\beta,-\beta,-\beta,-\mu,\gamma,\delta;1+\alpha-\mu,-\beta,-\beta;\frac{x}{x-1},\frac{w_1}{x-1},\frac{w_2}{x-1}\right],$$

......(4.2.1)

where

$$\left|\frac{w_1}{x-1}\right|<1,\left|\frac{w_2}{x-1}\right|<1,$$

$$D_x^{\mu}\left[x^{\alpha}(x-1)^{\beta}\left(1-\frac{w_1}{x-1}\right)^{-\gamma}(1-xw_2)^{-\delta}\right]$$

$$= A.F_N\left[\gamma,\delta,-\mu,-\beta,1+\alpha,-\beta;-\beta,1+\alpha-\mu,1+\alpha-\mu;\frac{w_1}{x-1},xw_2,\frac{x}{x-1}\right],$$

.....(4.2.2)

where

$$\left|\frac{w_1}{x-1}\right|<1,\ |xw_2|<1,$$

$$D_x^{\mu}\left[x^{\alpha}(x-1)^{\beta}(1-xw_1)^{-\gamma}(1-xw_2)^{-\delta}\right]$$

$$= AF_s\left[-\beta,1+\alpha,1+\alpha,-\mu,\gamma,\delta;1+\alpha-\mu,1+\alpha-\mu,1+\alpha-\mu;\frac{x}{x-1},xw_1,xw_2\right],$$

.......(4.2.3)

where

$$|xw_1|<1,|xw_2|<1,$$

$$D_x^{\mu}\left[x^{\alpha}(x-1)^{-\gamma}\left(1-\frac{xw_1}{x-1}-\frac{xw_2}{x-1}\right)^{-\beta}\right]$$

$$=B.F^{(3)}\left[\begin{matrix}\gamma & ::\beta,1+\alpha;-;-:-;-;-\mu; \\ 1+\alpha-\mu:: & \gamma\ ;-;-:-;-;-;\end{matrix}\ \frac{xw_1}{x-1},\frac{xw_2}{x-1},\frac{x}{x-1}\right],$$

.......(4.2.4)

where

$$\left|\frac{x(w_1+w_2)}{x-1}\right|<1,$$

$$D_x^\mu\left[x^\alpha(x-1)^\beta\left(1-\frac{w_1}{x-1}-\frac{w_2}{x-1}\right)^{-\gamma}\right]$$

$$=A.F^{(3)}\left[\begin{matrix}-\beta:: & \gamma;-;-:-;-; & -\mu; \\ -\ :: & -\beta;-;-:-;-; & 1+\alpha-\mu;\end{matrix}\ \frac{w_1}{x-1},\frac{w_2}{x-1},\frac{x}{x-1}\right], \quad ....(4.2.5)$$

where

$$\left|\frac{w_1+w_2}{x-1}\right|<1$$

and

$$D_x^\mu\left[x^\alpha(x-1)^\beta(1-xw_1-xw_2)^{-\gamma}\right]$$

$$=A.F^{(3)}\left[\begin{matrix}- & ::\ \gamma,1+\alpha; & -;-:-;-;-\mu,-\beta; \\ 1+\alpha-\mu:: & -;-; & -:-;\ -;\ -;\end{matrix}\ xw_1,xw_2,\frac{x}{x-1}\right],$$

.......(4.2.6)

where

$$|xw_1+xw_2|<1$$

and

$$A=x^{\alpha-\mu}(x-1)^\beta e^{-i\pi\mu}\frac{\Gamma(\mu-\alpha)}{\Gamma(-\alpha)},$$

$$B=x^{\alpha-\mu}(x-1)^{-\gamma} e^{-i\pi\mu}\frac{\Gamma(\mu-\alpha)}{\Gamma(-\alpha)}.$$

The functions F_G, F_N and F_S are the Saran's functions of three variables defined by (1.4.7), (1.4.9) and (1.4.10) respectively and $F^{(3)}[x,y,z]$ is the triple hypergeometric series defined by (1.4.11).

PROOFS:

To prove the results (4.2.1)-(4.2.6), we use the following results (Nishimato [51, 53, 56]):

$$\left(z^{\beta}\right)_{\alpha} = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}z^{\beta-\alpha} \quad ,\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right|<\infty \qquad \text{.......(4.2.7)}$$

$$\left((z-a)^{\beta}\right)_{\alpha} = e^{-i\pi\alpha}\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}(z-a)^{\beta-\alpha} \quad ,\left|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}\right|<\infty \qquad \text{.......(4.2.8)}$$

$$\left(u.v\right)_{\alpha} = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha+1)}{\Gamma(1+\alpha-n)\Gamma(n+1)}u_{\alpha-n}\,v_{n}\,. \qquad \text{.......(4.2.9)}$$

Proof of (4.2.1):

We know that

$$\left(1-x\right)^{-\gamma} = \sum_{n=0}^{\infty}\frac{(\gamma)^{n}}{n!}x^{n}, |x|<1,$$

Since the order of differentiation and summation is interchangeable under mentioned conditions, by using (4.2.9) thus, LHS of (4.2.1) gives.

$$= \sum_{k,m,n=0}^{\infty}\frac{(\gamma)_{m}(\delta)_{n}\,w_{1}^{m}\,w_{2}^{n}}{m!n!}\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)\Gamma(k+1)}\left(x^{\alpha}\right)_{\mu-k}\left((x-1)^{\beta-m-n}\right)_{k}$$

$$= \sum_{k,m,n=0}^{\infty} \frac{(\gamma)_m (\delta)_n w_1^m w_2^n}{k!m!n!} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)} \frac{\Gamma(\mu-k-\alpha)}{\Gamma(-\alpha)} \frac{\Gamma(k+m+n-\beta)}{\Gamma(m+n-\beta)}$$

$\times e^{-i\pi\mu} x^{\alpha-\mu+k} (x-1)^{\beta-m-n-k}$ [by using (4.2.7) and (4.2.8)].

$$= e^{-i\pi\mu} \frac{\Gamma(\mu-\alpha)}{\Gamma(-\alpha)} x^{\alpha-\mu} (x-1)^{\beta}$$

$$\times \sum_{k,m,n=0}^{\infty} \frac{(-\beta)_{k+m+n} (-\mu)_k (\gamma)_m (\delta)_n}{(1+\alpha-\mu)_k (-\beta)_{m+n} k!m!n!} \left(\frac{x}{x-1}\right)^k \left(\frac{w_1}{x-1}\right)^m \left(\frac{w_2}{x-1}\right)^n$$

which readily yields the RHS of (4.2.1). This complete the proof of (4.2.1). Results (4.2.2)-(4.2.6) can be proved similarly.

It is important to note that, the generalization of (4.2.1) and (4.2.3) can be obtained in the following form:

$$D_x^{\mu}\left[x^{\alpha}(x-1)^{\beta}\left(1-\frac{w_1}{x-1}\right)^{-\gamma_1}\left(1-\frac{w_2}{x-1}\right)^{-\gamma_2}\cdots\cdots\left(1-\frac{w_n}{x-1}\right)^{-\gamma_n}\right]$$

$$= A_{(1)}^{(1)} E_D^{(n+1)}\left[-\beta,-\mu,\gamma_1,\cdots,\gamma_n;1+\alpha-\mu,-\beta;\frac{x}{x-1},\frac{w_1}{x-1},\cdots,\frac{w_n}{x-1}\right],$$

....(4.2.10)

and

$$D_x^{\mu}\left[x^{\alpha}(x-1)^{\beta}(1-xw_1)^{-\gamma_1}\ldots\ldots(1-xw_n)^{-\gamma_1}\right]$$

$$= A_{(2)}^{(1)} E_D^{(n+1)}\left[-\beta,1+\alpha,-\mu,\gamma_1,\cdots,\gamma_n;1+\alpha-\mu;\frac{x}{x-1},xw_1,\ldots\ldots xw_n\right],\ldots.(4.2.11)$$

where ${}^{(k)}_{(1)}E_D^{(n)}$ and ${}^{(k)}_{(2)}E_D^{(n)}$ are the multiple hypergeometric series defined by (1.4.13) and (1.4.14).

4.3 LINEAR, DOUBLE AND MULTIPLE GENERATING FUNCTIONS

We consider the following elementary identities:

$$[(1-x)-t]^{-\lambda} = (1-t)^{-\lambda}\left(1-\frac{x}{1-t}\right)^{-\lambda} \qquad \text{.......(4.3.1)}$$

and

$$[1-(1-x)\,t]^{-\lambda} = (1-t)^{-\lambda}\left(1+\frac{xt}{1-t}\right)^{-\lambda}. \qquad \text{.......(4.3.2)}$$

Now, let us write (4.3.1) as

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^{-(\lambda+n)}\,t^n = (1-t)^{-\lambda}\left[1-\frac{x}{1-t}\right]^{-\lambda}, |t|<|1-x|. \qquad \text{.......(4.3.3)}$$

Replace x by $\frac{x(w_1+w_2)}{x-1}$, multiply both sides of (4.3.3) by $x^{\alpha}(x-1)^{-\beta}$ and operate then by the fractional derivative operator D_x^{μ}, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,t^n\,D_x^{\mu}\left[x^{\alpha}(x-1)^{-\beta}\left(1-\frac{x(w_1+w_2)}{x-1}\right)^{-(\lambda+n)}\right]$$

$$=(1-t)^{-\lambda}\,D_x^{\mu}\left[x^{\alpha}(x-1)^{-\beta}\left(1-\frac{x(w_1+w_2)}{(x-1)(1-t)}\right)^{-\lambda}\right] \qquad \text{.......(4.3.4)}$$

Now, in (4.3.4) using (4.2.4), we obtain the following linear generating function:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}\beta & ::\lambda+n,1+\alpha;\,-;-:-;-;-\mu; \\ 1+\alpha-\mu:: & \beta;\,-;-:-;-;-;\end{matrix}\,\frac{xw_1}{x-1},\frac{xw_2}{x-1},\frac{x}{x-1}\right]t^n$$

$$=(1-t)^{-\lambda}F^{(3)}\left[\begin{matrix}\beta & ::\lambda,1+\alpha;\,-;-:-;-;-\mu; \\ 1+\alpha-\mu:: & \beta;\,-;-:-;-;-;\end{matrix}\,\frac{xw_1}{(x-1)(1-t)},\frac{xw_2}{(x-1)(1-t)},\frac{x}{x-1}\right].$$

.......(4.3.5)

We adopt the analysis employed to obtain (4.3.5) and use results (4.2.5) and (4.2.6) respectively, to obtain the following two linear generating functions:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}-\beta::\lambda+n;-;-:-;-; & -\mu; \\ -::-\beta;-;-:-;-;1+\alpha-\mu; & \end{matrix}\,\frac{w_1}{x-1},\frac{w_2}{x-1},\frac{x}{x-1}\right]t^n$$

$$=(1-t)^{-\lambda}F^{(3)}\left[\begin{matrix}-\beta::\lambda;-;-:-;-; & -\mu; \\ -::-\beta;-;-:-;-;1+\alpha-\mu; & \end{matrix}\,\frac{w_1}{(x-1)(1-t)},\frac{w_2}{(x-1)(1-t)},\frac{x}{x-1}\right]$$

.......(4.3.6)

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}- & ::\lambda+n,1+\alpha;-;-:-;-;-\mu,-\beta; \\ 1+\alpha-\mu:: & -;-;-:-;-;-;\end{matrix}\,xw_1,xw_2,\frac{x}{x-1}\right]t^n$$

$$=(1-t)^{-\lambda}F^{(3)}\left[\begin{matrix}- & ::\lambda,1+\alpha;-;-:-;-;-\mu; \\ 1+\alpha-\mu:: & -;-;-:-;-,-;\end{matrix}\,\frac{xw_1}{1-t},\frac{xw_2}{1-t},\frac{x}{x-1}\right]. \quad ....(4.3.7)$$

Now, replace x by $\frac{w_1}{x-1},\frac{w_2}{x-1}$ respectively, replace t by t_1,t_2 *and* λ by λ_1,λ_2 respectively in (4.3.3). Then multiply the two equations, so obtained by each other, we have

$$= \sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m!n!} t_1^m t_2^n \left(1 - \frac{w_1}{x-1}\right)^{-(\lambda_1+m)} \left(1 - \frac{w_2}{x-1}\right)^{-(\lambda_2+n)}$$

$$= (1-t)^{-\lambda_1} (1-t_2)^{-\lambda_2} \left(1 - \frac{w_1}{(x-1)(1-t_1)}\right)^{-\lambda_1} \left(1 - \frac{w_2}{(x-1)(1-t_2)}\right)^{-\lambda_2} \qquad \text{.......(4.3.8)}$$

Multiply both sides of (4.3.8) by $x^{\alpha} (x-1)^{\beta}$ and then operate both sides by the fractional derivative operator D_x^{μ} and using (4.2.1), we obtain the following double generating function:

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m!n!} t_1^m t_2^n$$

$$F_G\left[-\beta,-\beta,-\beta,-\mu,\lambda_1+m,\lambda_2+n;1+\alpha-\mu,-\beta,-\beta;\frac{x}{x-1},\frac{w_1}{x-1},\frac{w_2}{x-1}\right]$$

$$= (1-t_1)^{-\lambda_1} (1-t_2)^{-\lambda_2}$$

$$F_G\left[-\beta,-\beta,-\beta,-\mu,\lambda_1,\lambda_2;1+\alpha-\mu,-\beta,-\beta;\frac{x}{x-1},\frac{w_1}{(x-1)(1-t_1)},\frac{w_2}{(x-1)(1-t_2)}\right].$$

$$\text{..... (4.3.9)}$$

The generalization of generating function (4.3.9) can be obtained in the following form:

$$\sum_{m_1,\cdots,m_n=0}^{\infty} \frac{(\lambda_1)_{m_1} \cdots\cdots (\lambda_n)_{m_n}}{m_1! \cdots\cdots m_n!} t_1^{m_1} \cdots\cdots t_n^{m_n}$$

$$\times {}^{(1)}_{(1)}E_D^{(n+1)}\left[-\beta,-\mu,\lambda_1+m_1,\cdots\cdots,\lambda_n+m_n;1+\alpha-\mu,-\beta;\frac{x}{x-1},\frac{w_1}{x-1},\cdots\cdots,\frac{w_n}{x-1}\right]$$

$$= (1-t_1)^{-\lambda_1} \cdots\cdots (1-t_n)^{-\lambda_n}$$

$$\times {}^{(1)}_{(1)}E_D^{(n+1)}\left[-\beta,-\mu,\lambda_1,\cdots,\lambda_n;1+\alpha-\mu,-\beta;\frac{x}{x-1},\frac{w_1}{(x-1)(1-t_1)},\cdots,\frac{w_n}{(x-1)(1-t_n)}\right].$$

$$\text{.......(4.3.10)}$$

Further, we adopt the analysis employed to obtain (4.3.9) and use results (4.2.2) and (4.2.3), we then obtain the following double generating functions:

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m!\,n!} t_1^m t_2^n$$

$$F_N\left[\lambda_1 + m,\ \lambda_2 + n, -\mu, -\beta, 1+\alpha, -\beta; -\beta, 1+\alpha-\mu, 1+\alpha-\mu, \frac{w_1}{x-1}, xw_2\,\frac{x}{x-1}\right]$$

$$= (1-t_1)^{-\lambda_1}\,(1-t_2)^{-\lambda_2}$$

$$F_N\left[\lambda_1, \lambda_2, -\mu, -\beta, 1+\alpha, -\beta; -\beta, 1+\alpha-\mu, 1+\alpha-\mu; \frac{w_1}{(x-1)(1-t_1)}, \frac{xw_2}{1-t_2}, \frac{x}{x-1}\right]$$
$$\ldots\ldots.(4.3.11)$$

and

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m!\,n!} t_1^m t_2^n$$

$$F_S\left[-\beta, 1+\alpha, 1+\alpha, -\mu, \lambda_1 + m, \lambda_2 + n;\ 1+\alpha-\mu,\ 1+\alpha-\mu,\ 1+\alpha-\mu;\ \frac{x}{x-1}, xw_1, xw_2\right]$$

$$= (1-t_1)^{-\lambda_1}\,(1-t_2)^{-\lambda_2}\,F_S\left[-\beta, 1+\alpha, 1+\alpha, -\mu, \lambda_1, \lambda_2; 1+\alpha-\mu, 1+\alpha-\mu,\ 1+\alpha-\mu;\ \frac{x}{x-1}, \frac{xw_1}{1-t}, \frac{xw_2}{1-t}\right]. \qquad \ldots\ldots(4.3.12)$$

The generalization of generating functions (4.3.12) can be obtained in the following form:

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\lambda_1)_{m_1}.....(\lambda_n)_{m_n}}{m_1!....m_n!} t_1^{m_1}.....t_n^{m_n}$$

$${}^{(1)}_{(2)}E_D^{(n+1)}\left[-\beta, 1+\alpha, -\mu, \lambda_1+m_1, ..., \lambda_n+m_n; 1+\alpha-\mu; \frac{x}{x-1}, xw_1, ..., xw_n \right]$$

$$=(1-t_1)^{-\lambda_1}....(1-t_n)^{-\lambda_n}$$

$${}^{(1)}_{(2)}E_D^{(n+1)}\left[-\beta, 1+\alpha, -\mu, \lambda_1, ..., \lambda_n; 1+\alpha-\mu; \frac{x}{x-1}, \frac{xw_1}{1-t}, ..., \frac{xw_n}{1-t} \right]. \qquad ......(4.3.13)$$

Now, we use the identity (4.3.2) to establish some more generating functions.

Let the identity (4.3.2) be expressed as:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-x)^n t^n = (1-t)^{-\lambda} \left(1 - \frac{xt}{t-1}\right)^{-\lambda}, |t| < |1-x|^{-1} \qquad ......(4.3.14)$$

Multiply both sides of (4.3.14) by $(1-x)^{-\rho}$ and replacing x by $\frac{x(w_1+w_2)}{x-1}$, we have

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(1 - \frac{x(w_1+w_2)}{x-1}\right)^{n-\rho} t^n = (1-t)^{-\lambda} \left(1 - \frac{x(w_1+w_2)}{x-1}\right)^{-\rho} \left(1 - \frac{tx(w_1+w_2)}{(x-1)(t-1)}\right)^{-\lambda}$$

$$......(4.3.15)$$

Now, multiply (4.3.15), on both sides, by $x^{\alpha}(x-1)^{-(\alpha+1)}$, operate by fractional derivative operator D_x^{μ} and using (4.2.4), we then obtain the following generating function:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F^{(3)} \left[\begin{matrix} \alpha+1 :: & \rho-n; & -;-:-;-;-\mu; \\ 1+\alpha-\mu :: & -; & -;-:-;-;-; \end{matrix} \frac{xw_1}{x-1}, \frac{xw_2}{x-1}, \frac{x}{x-1} \right] t^n$$

$$=(1-t)^{-\lambda}\, F_D^{(3)}\left[1+\alpha,-\mu,\rho,\lambda;1+\alpha-\mu;\ \frac{x(w_1+w_2)}{x-1},\frac{xt(w_1+w_2)}{(x-1)(t-1)}\right]. \quad \ldots\ldots(4.3.16)$$

where $F_D^{(3)}$ is the Lauricella function for n = 3, defined by (1.4.2).

Now, we adopt the analysis employed to obtain (4.3.16) and use the results (4.2.5) and (4.2.6) respectively, to obtain the following two linear generating relations:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}-\beta:: & \rho-n; & -;-:-;-;-\mu; \\ -:: & -\beta;-; & -:-;-;1+\alpha-\mu;\end{matrix}\ \frac{w_1}{x-1},\frac{w_2}{x-1},\frac{x}{x-1}\right]t^n$$

$$=(1-t)^{-\lambda}\, F_G\left[-\beta,-\beta,-\beta,-\mu,\rho,\lambda;1+\alpha-\mu,-\beta,-\beta;\right.$$

$$\left.\frac{x}{x-1},\frac{w_1+w_2}{x-1},\frac{t(w_1+w_2)}{(x-1)(t-1)}\right] \quad \ldots\ldots(4.3.17)$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}-:: & \rho-n,1+\alpha;-;-:-;-; & -\mu; \\ 1+\alpha-\mu:: & -;-;-:-;-; & -;\end{matrix}\ xw_1,xw_2,\frac{x}{x-1}\right]t^n$$

$$=(1-t)^{-\lambda}\ F_S\left[-\beta,1+\alpha,1+\alpha,-\mu,\rho,\lambda;1+\alpha-\mu,1+\alpha-\mu,1+\alpha-\mu;\right.$$

$$\left.\frac{x}{x-1},x(w_1+w_2),\frac{xt(w_1+w_2)}{t-1}\right]. \quad \ldots\ldots(4.3.18)$$

4.4 BILINEAR, DOUBLE AND MULTIPLE GENERATING FUNCTIONS

We consider the following identity {cf. [94], p.297}.

$$[(1-x)(1-y)-t]^{-\lambda}=(1-y)^{-\lambda}\left[\left(1-\frac{x}{1-t}\right)\left(1-\frac{y}{1-t}\right)-\frac{xyt}{(1-t)^2}\right]^{-\lambda} \quad \text{......(4.4.1)}$$

where

$$\left|\frac{t}{(1-x)(1-y)}\right|<1 \quad and \quad \left|\frac{xyt}{(1-x-t)(1-y-t)}\right|<1,$$

write (4.4.1) as

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^{-(\lambda+n)}(1-y)^{-(\lambda+n)}\,t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(1-\frac{x}{1-t}\right)^{-(\lambda+n)}\left(1-\frac{y}{1-t}\right)^{-(\lambda+n)}\left(\frac{xyt}{(1-t)^2}\right)^n. \quad \text{......(4.4.2)}$$

Now, in (4.4.2), we replace x, y, t and λ by $\frac{w_1}{x-1}, \frac{z_1}{y-1}, t_1$ *and* λ_1 respectively. Again replace x, y, t and λ by $\frac{w_2}{x-1}, \frac{z_2}{y-1}, t_2$ *and* λ_2 respectively. Then multiply two equations, so obtained, by each other, we get

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}\left(1-\frac{w_1}{x-1}\right)^{-(\lambda_1+m)}\left(1-\frac{w_2}{x-1}\right)^{-(\lambda_2+n)}$$

$$\times\left(1-\frac{z_1}{y-1}\right)^{-(\lambda_1+m)}\left(1-\frac{z_2}{y-1}\right)^{-(\lambda_2+n)} t_1^m\ t_2^n$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!\,n!}$$

$$\left(\frac{w_1 z_1 t_1}{(x-1)(y-1)(1-t_1)^2}\right)^m\left(\frac{w_2 z_2 t_2}{(x-1)(y-1)(1-t_2)^2}\right)^n\left(1-\frac{w_1}{(x-1)(1-t_1)}\right)^{-(\lambda_1+m)}$$

$$\times\left(1-\frac{w_2}{(x-1)(1-t_2)}\right)^{-(\lambda_2+n)}\left(1-\frac{z_1}{(y-1)(1-t_1)}\right)^{-(\lambda_1+m)}\left(1-\frac{z_2}{(y-1)(1-t_2)}\right)^{-(\lambda_2+n)}$$

......(4.4.3)

Now, multiply both sides of (4.4.3) by $x^\alpha(x-1)^\beta y^\gamma(y-1)^\delta$. Then operate both sides by fractional derivative operators D_x^μ *and* D_y^ν respectively and using (4.2.1), we arrive at the following double generating function:

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!n!}t_1^m\ t_2^n$$

$$F_G\left[-\beta,-\beta,-\beta,-\mu,\lambda_1+m,\lambda_2+n;1+\alpha-\mu,-\beta,-\beta;\frac{x}{x-1},\frac{w_1}{x-1},\frac{w_2}{x-1}\right]$$

$$F_G\left[-\delta,-\delta,-\delta,-\nu,\lambda_1+m,\lambda_2+n;1+\gamma-\nu,-\delta,-\delta;\frac{y}{y-1},\frac{z_1}{y-1},\frac{z_2}{y-1}\right]$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}$$

$$\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!n!}\left(\frac{w_1 z_1 t_1}{(x-1)(y-1)(1-t_1)^2}\right)^m\left(\frac{w_2 z_2 t_2}{(x-1)(y-1)(1-t_2)^2}\right)^n$$

$$F_G\Bigg[\; m+n-\beta, m+n-\beta, m+n-\beta, -\mu, \lambda_1+m, \lambda_2+n;$$

$$1+\alpha-\mu, m+n-\beta, m+n-\beta; \frac{x}{x-1}, \frac{w_1}{(x-1)(1-t_1)}, \frac{w_2}{(x-1)(1-t_2)}\Bigg]$$

$$F_G\Bigg[\; m+n-\delta, m+n-\delta, m+n-\delta, -\nu, \lambda_1+m, \lambda_2+n;$$

$$1+\gamma-\nu, m+n-\delta, m+n-\delta; \frac{y}{y-1}, \frac{z_1}{(y-1)(1-t_1)}, \frac{z_2}{(y-1)(1-t_2)}\Bigg]. \quad ......(4.4.4)$$

The generalization of (4.4.4) can be obtained in the following form:

$$\sum_{m_1,\cdots,m_n=0}^{\infty} \frac{(\lambda_1)_{m_1}\cdots\cdots(\lambda_n)_{m_n}}{m_1!\cdots\cdots m_n!} t_1^{m_1}\cdots\cdots t_n^{m_n}$$

$${}^{(1)}_{(1)}E_D^{(n+1)}\left[-\beta, -\mu, \lambda_1+m_1, \cdots, \lambda_n+m_n; 1+\alpha-\mu, -\beta; \frac{x}{x-1}, \frac{w_1}{x-1}, \cdots, \frac{w_n}{x-1}\right]$$

$${}^{(1)}_{(1)}E_D^{(n+1)}\left[-\delta, -\nu, \lambda_1+m_1, \cdots\cdots, \lambda_n+m_n; 1+\gamma-\nu, -\delta; \frac{y}{y-1}, \frac{z_1}{y-1}, \cdots, \frac{z_n}{y-1}\right]$$

$$=(1-t_1)^{-\lambda_1}\cdots(1-t_n)^{-\lambda_n}\left(\frac{w_1 z_1 t_1}{(x-1)(y-1)(1-t_1)^2}\right)^{m_1}\cdots\left(\frac{w_2 z_2 t_2}{(x-1)(y-1)(1-t_n)^2}\right)^{m_n}$$

$${}^{(1)}_{(1)}E_D^{(n+1)}\Bigg[\; m_1+\cdots+m_n-\beta, -\mu, \lambda_1+m_1, \cdots, \lambda_n+m_n; 1+\alpha-\mu, m_1+\cdots+m_n-\beta;$$

$$\frac{x}{x-1}, \frac{w_1}{(x-1)(1-t_1)}, \cdots, \frac{w_n}{(x-1)(1-t_n)}\Bigg]$$

$$ {}^{(1)}_{(1)}E_D^{(n+1)}\Bigg[\; m_1+\cdots+m_n-\delta,-\nu,\lambda_1+m_1,\cdots,\lambda_n+m_n\,; $$

$$ 1+\gamma-\nu,m_1+\cdots+m_n-\delta;\frac{y}{y-1},\frac{z_1}{(y-1)(1-t_1)},\cdots,\frac{z_n}{(y-1)(1-t_n)}\Bigg] \qquad \ldots\ldots(4.4.5) $$

Now, we adopt the analysis employed to obtain (4.4.4) and use results (4.2.2) and (4.2.3), we then arrive at the following two double generating functions:

$$ \sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!n!}t_1^m\,t_2^n $$

$$ F_N\left[\lambda_1+m,\ \lambda_2+n,-\mu,-\beta,1+\alpha,-\beta;-\beta,1+\alpha-\mu,1+\alpha-\mu,\frac{w_1}{x-1},xw_2\frac{x}{x-1}\right] $$

$$ F_N\left[\lambda_1+m,\ \lambda_2+n,-\nu,-\delta,1+\gamma,-\delta;-\delta,1+\gamma-\nu,1+\gamma-\nu,\frac{z_1}{y-1},\frac{z_2}{y-1}\frac{y}{y-1}\right] $$

$$ =(1-t_1)^{-\lambda_1}\,(1-t_1)^{-\lambda_2}\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!n!} $$

$$ \left(\frac{w_1\,z_1t_1}{(x-1)(y-1)(1-t_1)^2}\right)^m\left(\frac{xyw_2\,z_2t_2}{(1-t_2)^2}\right)^n\frac{(1+\alpha)_n\,(1+\gamma)_n}{(1+\alpha-\mu)_n\,(1+\gamma-\nu)_n} $$

$$ F_N\Bigg[\;\lambda_1+m,\lambda_2+n,-\mu,m-\beta,1+\alpha+n,m-\beta;m-\beta,1+\alpha+n-\mu,1+\alpha+n-\mu; $$

$$ \frac{w_1}{(x-1)(1-t_1)},\frac{xw_2}{1-t_2},\frac{x}{x-1}\Bigg] $$

$$ F_N\Bigg[\;\lambda_1+m,\lambda_2+n,-\nu,m-\delta,1+\gamma+n,m-\delta;m-\delta,1+\gamma+n-\nu,1+\gamma+n-\nu; $$

$$ \frac{z_1}{(y-1)(1-t_1)},\frac{yz_2}{1-t_2},\frac{y}{y-1}\Bigg] \qquad \ldots\ldots(4.4.6) $$

and

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m!\,n!} t_1^m t_2^n$$

$$F_S\left[-\beta, 1+\alpha, 1+\alpha, -\mu, \lambda_1+m, \lambda_2+n;\ 1+\alpha-\mu,\ 1+\alpha-\mu, 1+\alpha-\mu;\ \frac{x}{x-1}, xw_1, xw_2\right]$$

$$F_S\left[-\delta, 1+\gamma, 1+\gamma, -\nu, \lambda_1+m, \lambda_2+n;\ 1+\gamma-\nu,\ 1+\gamma-\nu, 1+\gamma-\nu;\ \frac{y}{y-1}, yz_1, yz_2\right]$$

$$=(1-t_1)^{-\lambda_1}\ (1-t_1)^{-\lambda_2} \sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m!\,n!}$$

$$\left(\frac{xyw_1\, z_1 t_1}{(1-t_1)^2}\right)^m \left(\frac{xyw_2\, z_2 t_2}{(1-t_2)^2}\right)^n \frac{(1+\alpha)_{m+n}\ (1+\gamma)_{m+n}}{(1+\alpha-\mu)_{m+n}\ (1+\gamma-\nu)_{m+n}}$$

$$F_S\left[-\beta, 1+\alpha+m+n, 1+\alpha+m+n, -\mu, \lambda_1+m, \lambda_2+n;\right.$$

$$\left. 1+\alpha+m+n-\mu,\ 1+\alpha+m+n-\mu, 1+\alpha+m+n-\mu; \frac{x}{x-1}, \frac{xw_1}{1-t_1}, \frac{xw_2}{1-t_2}\right]$$

$$F_S\left[-\delta, 1+\gamma+m+n, 1+\gamma+m+n, -\nu, \lambda_1+m, \lambda_2+n;\right.$$

$$\left. 1+\gamma+m+n-\nu,\ 1+\gamma+m+n-\nu, 1+\gamma+m+n-\nu; \frac{y}{y-1}, \frac{yz_1}{1-t_1}, \frac{yz_2}{1-t_2}\right].$$

......(4.4.7)

The generalization of (4.4.7) can be obtained in the following form:

$$\sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\lambda_1)_{m_1}\dots(\lambda_n)_{m_n}}{m_1!\dots m_n!} t_1^{m_1}\dots t_n^{m_n}$$

$${}^{(1)}_{(2)}E_D^{(n+1)}\left[-\beta, 1+\alpha, -\mu, \lambda_1+m_1,\dots,\lambda_n+m_n; 1+\alpha-\mu;\ \frac{x}{x-1}, xw_1,\dots,xw_n\right]$$

$${}^{(1)}_{(2)}E_D^{(n+1)}\left[-\delta, 1+\gamma, -\nu, \lambda_1+m_n,\dots,\lambda_n+m_n; 1+\gamma-\nu; \frac{y}{y-1}, yz_1,\dots,yz_n\right]$$

$$=(1-t_1)^{-\lambda_1}\dots(1-t_n)^{-\lambda_n} \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\lambda_1)_{m_1}\dots(\lambda_n)_{m_n}}{m_1!\dots m_n!}$$

$$\left(\frac{xyw_1 z_1 t_1}{(1-t_1)^2}\right)^{m_1}\dots\left(\frac{xyw_n z_n t_n}{(1-t_n)^2}\right)^{m_n} \frac{(1+\alpha)_{m_1+\dots+m_n}(1+\gamma)_{m_1+\dots+m_n}}{(1+\alpha-\mu)_{m_1+\dots+m_n}(1+\gamma-\nu)_{m_1+\dots+m_n}}$$

$${}^{(1)}_{(2)}E_D^{(n+1)}\left[-\beta, 1+\alpha+m_1+\dots+m_n, -\mu, \lambda_1+m_1,\dots,\lambda_n+m_n;\right.$$

$$\left.1+\alpha+m_1+\dots+m_n-\mu; \frac{x}{x-1}, \frac{xw_1}{1-t_1},\dots,\frac{xw_n}{1-t_n}\right]$$

$${}^{(1)}_{(2)}E_D^{(n+1)}\left[-\delta, 1+\gamma+m_1+\dots+m_n, -\nu, \lambda_1+m_1,\dots,\lambda_n+m_n;\right.$$

$$\left.1+\gamma+m_1+\dots+m_n-\nu; \frac{y}{y-1}, \frac{yz_1}{1-t_1},\dots,\frac{yz_n}{1-t_n}\right]. \qquad \dots\dots(4.4.8)$$

Now, in (4.4.2), we replace x and y by $\frac{x(w_1+w_2)}{x-1}$ *and* $\frac{y(z_1+z_2)}{y-1}$ respectively and then multiply both of it by $x^{\alpha}\ y^{\beta}\ (x-1)^{-\gamma}\ (y-1)^{-\delta}$ and then operate by D_x^{μ} *and* D_y^{ν} for x and y respectively and using (4.2.4), we obtain the following bilinear generating function:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}\gamma & ::\lambda+n,\ 1+\alpha\ ;-;-:-;-;-\mu; \\ 1+\alpha-\mu:: & \gamma\qquad ;-;-:-;-;\ -\ ;\end{matrix}\ \frac{xw_1}{x-1},\frac{xw_2}{x-1},\frac{x}{x-1}\right]$$

$$F^{(3)}\left[\begin{matrix}\delta & ::\lambda+n,\ 1+\beta\ ;-;-:-;-;-\nu; \\ 1+\beta-\nu:: & \delta\qquad ;-;-:-;-;\ -\ ;\end{matrix}\ \frac{yz_1}{y-1},\frac{yz_2}{y-1},\frac{y}{y-1}\right]t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{xyt(w_1+w_2)(z_1+z_2)}{(x-1)(y-1)(1-t)^2}\right)^n\frac{(1+\alpha)_n\,(1+\beta)_n}{(1+\alpha-\mu)_n\,(1+\beta-\nu)_n}$$

$$F^{(3)}\left[\begin{matrix}\gamma+n & ::\lambda+n,1+\alpha+n;-;-:-;-;-\mu; \\ 1+\alpha+n-\mu:: & \gamma+n\qquad ;-;-:-;-;\ -\ ;\end{matrix}\ \frac{xw_1}{(x-1)(1-t)},\frac{xw_2}{(x-1)(1-t)},\frac{x}{x-1}\right]$$

$$F^{(3)}\left[\begin{matrix}\delta+n & ::\lambda+n,1+\beta+n;-;-:-;-;-\nu; \\ 1+\beta+n-\nu:: & \delta+n\qquad ;-;-:-;-;\ -\ ;\end{matrix}\ \frac{yz_1}{(y-1)(1-t)},\frac{yz_2}{(y-1)(1-t)},\frac{y}{y-1}\right]$$

......(4.4.9)

A similar interesting bilinear generating functions, given below are obtained when we using the same method employed to obtain (4.4.9) with the help of the results (4.2.5) and (4.2.6).

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}-\gamma::\lambda+n;-;-:-;-; & -\mu & ; \\ -\ ::\ -\gamma\ ;-;-:-;-; & 1+\alpha-\mu & ;\end{matrix}\ \frac{w_1}{x-1},\frac{w_2}{x-1},\frac{x}{x-1}\right]$$

$$F^{(3)}\left[\begin{matrix}-\delta::\lambda+n;-;-:-;-; & -\nu & ; \\ -\ ::\ -\delta\ ;-;-:- & ;\ 1+\beta-\nu & ;\end{matrix}\ \frac{z_1}{y-1},\frac{z_2}{y-1},\frac{y}{y-1}\right]t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{(w_1+w_2)(z_1+z_2)t}{(1-t)^2(x-1)(y-1)}\right)^n$$

$$F^{(3)}\left[\begin{matrix}n-\gamma::\lambda+n;-;-:-;-; & -\mu & ; \\ -\ ::n-\gamma;-;-:-;-; & 1+\alpha-\mu & ;\end{matrix}\ \frac{w_1}{(x-1)(1-t)},\frac{w_2}{(x-1)(1-t)},\frac{x}{x-1}\right]$$

$$F^{(3)}\left[\begin{array}{l} n-\delta::\lambda+n;-;-:-;-;\quad -\nu\quad ;\dfrac{z_1}{(y-1)(1-t)},\dfrac{z_2}{(y-1)(1-t)},\dfrac{y}{y-1} \\ -\ ::n-\delta;-;-:-;-;1+\beta-\nu\ ; \end{array}\right]$$

......(4.4.10)

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{array}{l} -\quad ::\lambda+n,1+\alpha;-;-:-;-;-\mu,-\gamma; \\ 1+\alpha-\mu::\quad -\quad ;-;-:-;-;\quad -\quad ; \end{array} xw_1, xw_2, \frac{x}{x-1}\right]$$

$$F^{(3)}\left[\begin{array}{l} -\quad ::\lambda+n,1+\beta;-;-:-;-;-\nu,-\delta; \\ 1+\beta-\nu::\quad -\quad ;-;-:-;-;\quad -\quad ; \end{array} yz_1, yz_2, \frac{y}{y-1}\right]t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{(w_1+w_2)(z_1+z_2)xyt}{(1-t)^2}\right)^n\frac{(1+\alpha)_n\,(1+\beta)_n}{(1+\alpha-\mu)_n\,(1+\beta-\nu)_n}$$

$$F^{(3)}\left[\begin{array}{l} -\quad ::\lambda+n,1+\alpha+n;-;-:-;-;-\mu,-\gamma; \\ 1+\alpha+n-\mu::\quad -\quad ;-;-:-;-;\quad -\quad ; \end{array}\frac{xw_1}{x-1},\frac{xw_2}{x-1},\frac{x}{x-1}\right]$$

$$F^{(3)}\left[\begin{array}{l} -\quad ::\lambda+n,1+\beta+n;-;-:-;-;\nu,-\delta; \\ 1+\beta+n-\nu::\quad -\quad ;-;-:-;-;\quad -\quad ; \end{array}\frac{yz_1}{y-1},\frac{yz_2}{y-1},\frac{y}{y-1}\right].$$

......(4.4.11)

4.5 BILATERAL GENERATING FUNCTIONS

In this section, we employing the linear generating functions [discussed in the section (4.3)], to establish some bilateral generating functions.

We replace t in (4.3.5) by $t(1-y)$, multiply both sides of it by y^{γ} and operate by D_y^{ν} (for the variable y), we obtain

$$D_y^{\nu}\left(\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}\beta & ::\lambda+n,1+\alpha;\ -;-:-;-;-\mu; \\ 1+\alpha-\mu:: & \beta\ ;-;-:-;-;-;\end{matrix}\ \frac{xw_1}{x-1},\frac{xw_2}{x-1},\frac{x}{x-1}\right]y^{\gamma}(1-y)^n t^n\right)$$

$$=D_y^{\nu}\left(F^{(3)}\left[\begin{matrix}\beta & ::\lambda,1+\alpha;\ -;-:-;-;-\mu; \\ 1+\alpha-\mu:: & \beta\ ;-;-:-;-;-;\end{matrix}\ \frac{xw_1}{(x-1)(1-t)},\frac{xw_2}{(x-1)(1-t)},\frac{x}{x-1}\right]\right.$$

$$\left. y^{\gamma}[1-t(1-y)]^{-\lambda}\right) \quad ......(4.5.1)$$

Now, by using (4.2.7)–(4.2.9) and employing some usual calculations, we arrive at the following bilateral generating function:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}\beta & ::\lambda+n,1+\alpha;\ -;-:-;-;-\mu; \\ 1+\alpha-\mu:: & \beta\ ;-;-:-;-;-;\end{matrix}\ \frac{xw_1}{x-1},\frac{xw_2}{x-1},\frac{x}{x-1}\right]$$

$${}_2F_1[-n,1+\gamma;1+\gamma-\nu;y]t^n$$

$$=(1-t)^{-\lambda}\sum_{p=0}^{\infty}\frac{(\lambda)_p(1+\alpha)_p}{(1+\alpha-\mu)_p\,p!}\left(\frac{xw_1}{(x-1)(1-t)}\right)^p$$

$$F^{(3)}\left[\begin{matrix}-:: & \beta+p;-;\lambda+p: & 1+\alpha+p;-\mu;1+\gamma; \\ -:: & 1+\alpha-\mu+p;-;-: & \beta+p\ ;-;1+\gamma-\nu;\end{matrix}\ \frac{xw_2}{(x-1)(1-t)},\frac{x}{x-1},\frac{-yt}{1-t}\right].$$

. ...(4.5.2)

Now, the analysis that was used for obtaining (4.5.2), is employed again with (4.3.6) and (4.3.7) to obtain the following two bilateral generating functions:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{matrix}-\beta::\lambda+n;\ -;-: & -;\ -;-\mu; \\ -\ ::\ -\beta\ ;-;\ -:-; & -;1+\alpha-\mu;\end{matrix}\ \frac{w_1}{x-1},\frac{w_2}{x-1},\frac{x}{x-1}\right]$$

$${}_2F_1[-n,1+\gamma;1+\gamma-\nu;y]t^n$$

$$=(1-t)^{-\lambda}\sum_{p=0}^{\infty}\frac{(\lambda)_p}{p!}\left(\frac{w_1}{(x-1)(1-t)}\right)^p$$

$$F_k\left[\begin{array}{l}1+\gamma,-\beta+p,-\beta+p,\lambda+p,-\mu,\lambda+p;1+\gamma-v,1+\alpha-\mu,-\beta+p;\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{-yt}{1-t},\frac{x}{x-1},\frac{w_2}{(x-1)(1-t)}\end{array}\right]. \quad \ldots\ldots(4.5.3)$$

where F_k is the Saran's function defined by (1.4.8)

and

$$\sum_{p=0}^{\infty}\frac{(\lambda)_n}{n!}F^{(3)}\left[\begin{array}{c}-\ ::\lambda+n,1+\alpha\ ;-;-:\ -;-;-\mu,-\beta;\\ 1+\alpha-\mu\ ::\ -\ ;\ -;-:\ -;-;-;\end{array}xw_1,xw_2\frac{x}{x-1}\right]$$

$${}_2F_1[-n,\gamma+1;\gamma+1-v;y]t^n=(1-t)^{-\lambda}\sum_{p=0}^{\infty}\frac{(\lambda)_p(1+\alpha)_p}{(1+\alpha-\mu)_p\,p!}\left(\frac{xw_1}{1-t}\right)^p$$

$$F_N\left[\begin{array}{l}1+\gamma,-\beta,1+\alpha+p,\lambda+p,-\mu,\lambda+p;1+\gamma-v,1+\alpha-\mu+p,1+\alpha-\mu+p;\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{-yt}{1-t},\frac{x}{x-1},\frac{xw_2}{1-t}\end{array}\right]. \quad \ldots\ldots(4.5.4)$$

Further, if in (4.3.9), we replace t_1 *and* t_2 by $t_1(1-\eta_1 y)$ and $t_2(1-\eta_2 y)$ respectively, such that $|\eta_i|<1, i=1,2$. Multiply both sides by y^γ and then operate by D_y^v (for the variable y), we obtain

$$D_y^v\left(\sum_{m,n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!n!}t_1^m t_2^n(1-\eta_1 y)^m(1-\eta_2 y)^n\right.$$

$$\left.\times F_G\left[-\beta,-\beta,-\beta,-\mu,\lambda_1+m,\lambda_2+n;1+\alpha-\mu,-\beta,-\beta;\frac{x}{x-1},\frac{w_1}{x-1},\frac{w_2}{x-1}\right]\right)$$

$$= D_y^v\left(\left[y^r\left[1-t_1(1-\eta_1 y)\right]^{-\lambda_1}\left[1-t_2(1-\eta_2 y)\right]^{-\lambda_2}\right]\right.$$

$$\left.\times F_G\left[-\beta,-\beta,-\beta,-\mu,\lambda_1,\lambda_2;1+\alpha-\mu,-\beta,-\beta;\frac{x}{x-1},\frac{w_1}{(x-1)(1-t_1)},\frac{w_2}{(x-1)(1-t_2)}\right]\right)$$

......(4.5.5)

Now, by using (4.2.7)-(4.2.9) and employing some usual calculations, we arrive at the following bilateral generating function:

$$\sum_{n=0}^{\infty}\frac{(\lambda_1)_m(\lambda_2)_n}{m!n!}t_1^m t_2^n$$

$$F_G\left[-\beta,-\beta,-\beta,-\mu,\lambda_1+m,\lambda_2+n;1+\alpha-\mu,-\beta,-\beta;\frac{x}{x-1},\frac{w_1}{x-1},\frac{w_2}{x-1}\right]$$

$$F_1\left[1+\gamma,-m,-n;1+\gamma-v;\eta_1 y,\eta_2 y\right]$$

$$=(1-t_1)^{-\lambda_1}(1-t_2)^{-\lambda_2}\sum_{q,r=0}^{\infty}\frac{(\lambda_1)_q(\lambda_2)_r}{q!r!}\left(\frac{w_1}{(x-1)(1-t_1)}\right)^q\left(\frac{w_2}{(x-1)(1-t_2)}\right)^r$$

$${}_2F_1\left[-\beta+q+r,-\mu;1+\alpha-\mu;\frac{x}{x-1}\right]$$

$$F_1\left[1+\gamma,\lambda_1+q,\lambda_2+r;1+\gamma-v;\frac{t_1\eta_1 y}{t_1-1},\frac{t_2\eta_2 y}{t_2-1}\right]. \qquad ......(4.5.6)$$

Further, form linear generating functions (4.3.11) and (4.3.12), we obtain the following two bilateral generating functions. The method is the same as used for obtained (4.5.6)

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m!n!} t_1^m t_2^n$$

$$F_N\left[\lambda_1+m, \lambda_2+n, -\mu, -\beta, 1+\alpha, -\beta; -\beta, 1+\alpha-\mu, 1+\alpha-\mu; \frac{w_1}{x-1}, xw_2, \frac{x}{x-1}\right]$$

$$F_1\left[1+\gamma, -m, -n; 1+\gamma-v; \eta_1 y, \eta_2 y\right]$$

$$=(1-t_1)^{-\lambda_1}\ (1-t_2)^{-\lambda_2} \sum_{p,q=0}^{\infty} \frac{(\lambda_1)_p (\lambda_2)_q (1+\alpha)_q}{(1+\alpha-\mu)_q\, p!q!} \left(\frac{w_1}{(x-1)(1-t_1)}\right)^p \left(\frac{w_2}{(x-1)(1-t_2)}\right)^q$$

$${}_2F_1\left[-\beta+p, -\mu; 1+\alpha-\mu; \frac{x}{x-1}\right]$$

$$F_1\left[1+\gamma, \lambda_1+p, \lambda_2+q; 1+\gamma-v; \frac{t_1\eta_1 y}{t_1-1}, \frac{t_2\eta_2 y}{t_2-1}\right] \qquad \ldots\ldots(4.5.7)$$

and

$$\sum_{m,n=0}^{\infty} \frac{(\lambda_1)_m (\lambda_2)_n}{m!n!} t_1^m t_2^n$$

$$F_S\left[-\beta, 1+\alpha, 1+\alpha, -\mu, \lambda_1+m, \lambda_2+n; 1+\alpha-\mu, 1+\alpha-\mu, 1+\alpha-\mu; \frac{x}{x-1}, xw_1, xw_2\right]$$

$$F_1\left[1+\gamma, -m, -n; 1+\gamma-v; \eta_1 y, \eta_2 y\right]$$

$$=(1-t_1)^{-\lambda_1}\ (1-t_2)^{-\lambda_2} \sum_{q,r=0}^{\infty} \frac{(\lambda_1)_q (\lambda_2)_r (1+\alpha)_{q+r}}{(1+\alpha-\mu)_{q+r}\, q!r!} \left(\frac{xw_1}{1-t_1}\right)^q \left(\frac{xw_2}{1-t_2}\right)^r$$

$${}_2F_1\left[-\beta, -\mu; 1+\alpha-\mu; \frac{x}{x-1}\right]$$

$$F_1\left[1+\gamma, \lambda_1+q, \lambda_2+r; 1+\gamma-v; \frac{t_1\eta_1 y}{t_1-1}, \frac{t_2\eta_2 y}{t_2-1}\right]. \qquad \ldots\ldots(4.5.8)$$

4.6 SPECIAL CASES

Now, we mention some interesting special cases of our results in section [4.3].

On taking $w_2 = 0$, $\beta = \alpha + 1$ in (4.3.5) replace $\frac{x}{x-1}$, by y and w_1 by $\frac{z}{y}$, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_1 \left[1+\alpha, \lambda+n, -\mu; 1+\alpha-\mu; z, y\right] t^n$$

$$= (1-t)^{-\lambda} F_1 \left[1+\alpha,\ \lambda, -\mu;\ 1+\alpha-\mu; \frac{z}{1-t}, y \right], \qquad \text{......(4.6.1)}$$

which for y → 0 reduces to a know result {[94], p.292 (6)}.

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_2F_1 \left[\lambda+n, 1+\alpha; 1+\alpha-\mu; z\right] t^n$$

$$= (1-t)^{-\lambda}\ {}_2F_1 \left[\lambda,\ 1+\alpha; 1+\alpha-\mu; \frac{z}{1-t} \right]. \qquad \text{......(4.6.2)}$$

For $W_2 = 0$, (4.3.6) and (4.3.7) reduces respectively to the following two linear generating functions:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_2 \left[-\beta, \lambda+n, -\mu; -\beta, 1+\alpha-\mu; \frac{w_1}{x-1}, \frac{x}{x-1} \right] t^n$$

$$= (1-t)^{-\lambda} F_2 \left[-\beta, \lambda+n, -\mu;\ -\beta, 1+\alpha-\mu; \frac{w_1}{(x-1)(1-t)}, \frac{x}{x-1} \right] \qquad \text{......(4.6.3)}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_3\left[\lambda+n,-\mu,1+\alpha,-\beta;1+\alpha-\mu;xw_1,\frac{x}{x-1}\right]t^n$$

$$=(1-t)^{-\lambda}F_3\left[\lambda,-\mu,1+\alpha,-\beta;1+\alpha-\mu;\frac{xw_1}{1-t},\frac{x}{x-1}\right]. \qquad ......(4.6.4)$$

Now, if in (4.3.16), we put $w_2=0$, replace $\frac{x}{x-1}$ by y and w_1 by $\frac{z}{y}$, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1\left[\alpha+1,-\mu,\rho-n;1+\alpha-\mu;y,z\right]t^n$$

$$=(1-t)^{-\lambda}F_D^{(3)}\left[\alpha+1,-\mu,\rho,\lambda;1+\alpha-\mu;y,z,\frac{zt}{t-1}\right], \qquad ......(4.6.5)$$

which for y → 0 reduces to a know result {[94] p.151 (44)}.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}{}_2F_1\left[\rho-n,\alpha+1;1+\alpha-\mu;z\right]t^n$$

$$=(1-t)^{-\lambda}F_1\left[\alpha+1,\rho,\lambda;1+\alpha-\mu;z,\frac{zt}{t-1}\right]. \qquad ......(4.6.6)$$

For $w_2=0$, (4.3.17) and (4.3.18) reduces to the following results:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_2\left[-\beta,\rho-n,-\mu;-\beta,1+\alpha-\mu;\frac{w_1}{x-1},\frac{x}{x-1}\right]t^n$$

$$=(1-t)^{-\lambda}F_G\left[-\beta,-\beta,-\beta,-\mu,\rho,\lambda;1+\alpha-\mu,-\beta,-\beta;\right.$$

$$\left.\frac{x}{x-1},\frac{w_1}{x-1},\frac{w_1 t}{(x-1)(1-t)}\right] \qquad ......(4.6.7)$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_3\left[-\beta,\rho-n,-\mu,1+\alpha;1+\alpha-\mu;\frac{x}{x-1},xw_1\right]t^n$$

$$=(1-t)^{-\lambda}F_S\left[-\beta,1+\alpha,1+\alpha,-\mu,\rho,\lambda;1+\alpha-\mu,1+\alpha-\mu,1+\alpha-\mu;\right.$$

$$\left.\frac{x}{x-1},xw_1,\frac{xtw_1}{t-1}\right]. \qquad \text{......(4.6.8)}$$

where $F_1, F_2 \, and \, F_3$ are Appell's functions defined by (1.3.4), (1.3.5) and (1.3.6) respectively.

Now, we mention some other interesting special cases of our results (4.4.9), (4.4.10) and (4.4.11) in the section (4.4).

On taking $w_2=z_2=0$, $\gamma=1+\alpha, \delta=1+\beta$ and replacing

$\frac{x}{x-1}, \frac{y}{y-1}, w_1$ and $z_1 \; by \, y, x, \frac{z}{y} \; and \frac{w}{x}$ respectively, (4.4.9) reduces to

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_1\left[1+\alpha,\lambda+n,-\mu;1+\alpha-\mu;z,y\right]$$

$$F_1\left[1+\beta,\lambda+n,-\nu;1+\beta-\nu;w,x\right]t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{wzt}{(1-t)^2}\right)^n\frac{(1+\alpha)_n\,(1+\beta)_n}{(1+\alpha-\mu)_n\,(1+\beta-\nu)_n}$$

$$F_1\left[\alpha+n+1,\lambda+n,-\mu;1+\alpha+n-\mu;\frac{z}{1-t},y\right]$$

$$F_1\left[\beta+n+1,\lambda+n,-\nu;1+\beta+n-\nu;\frac{w}{1-t},x\right]. \qquad \text{......(4.6.9)}$$

Further, letting x, y → 0 in (4.6.9), we get a know result due to Meixner {[48], p.345 (19c)}.

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1\left[\;1+\alpha,\lambda+n;1+\alpha-\mu;z\;\right]{}_2F_1\left[\;1+\beta,\lambda+n\;;1+\beta-\nu;w\;\right]t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{wzt}{(1-t)^2}\right)^n\frac{(1+\alpha)_n\,(1+\beta)_n}{(1+\alpha-\mu)_n\,(1+\beta-\nu)_n}$$

$${}_2F_1\left[\alpha+n+1,\lambda+n\;;1+\alpha+n-\mu;\frac{z}{1-t}\right]$$

$${}_2F_1\left[\beta+n+1,\lambda+n;1+\beta+n-\nu;\frac{w}{1-t}\right]. \qquad ......(4.6.10)$$

For $w_2 = z_2 = 0$, (4.4.10) and (4.4.11) reduces respectively to the following two bilinear generating functions:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_2\left[-\gamma,\lambda+n,-\mu;-\gamma,1+\alpha-\mu;\frac{w_1}{x-1},\frac{x}{x-1}\right]$$

$$F_2\left[-\delta,\lambda+n,-\nu;-\delta,1+\beta-\nu;\frac{z_1}{y-1},\frac{y}{y-1}\right]t^n$$

$$=(1-t)^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{w_1z_1t}{(x-1)(y-1)(1-t)^2}\right)^n$$

$$F_2\left[n-\gamma,\lambda+n,-\mu;n-\gamma,1+\alpha-\mu;\frac{w_1}{(x-1)\,(1-t)},\frac{x}{x-1}\right]$$

$$F_2\left[n-\delta,\lambda+n,-\nu;n-\delta,1+\beta-\nu;\frac{z_1}{(y-1)(1-t)},\frac{y}{y-1}\right] \qquad ......(4.6.11)$$

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_3 \left[\lambda + n, -\mu, 1+\alpha, -\gamma; 1+\alpha-\mu; xw_1, \frac{x}{x-1}\right]$$

$$F_3 \left[\lambda + n, -\nu, 1+\beta, -\delta; 1+\beta-\nu; yz_1, \frac{y}{y-1}\right] t^n$$

$$= (1-t)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\frac{xytw_1z_1}{(1-t)^2}\right)^n \frac{(1+\alpha)_n (1+\beta)_n}{(1+\alpha-\mu)_n (1+\beta-\nu)_n}$$

$$F_3 \left[\lambda + n, -\mu, 1+\alpha+n, -\gamma; 1+\alpha+n-\mu; \frac{xw_1}{1-t}, \frac{x}{x-1}\right]$$

$$F_3 \left[\lambda + n, -\nu, 1+\beta+n, -\delta; 1+\beta+n-\nu; \frac{yz_1}{1-t}, \frac{y}{y-1}\right]. \qquad \text{......(4.6.12)}$$

Finally, we mention some interesting particular cases of our results (4.5.2), (4.5.3) and (4.5.4) in the section (4.5)

On taking $w_2 = 0, \gamma = \alpha + 1$ in (4.5.2), replacing $\frac{x}{x-1}$ by y and w_1 $by \frac{z}{y}$ and letting y → 0, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_2F_1 [\lambda + n, 1+\alpha; 1+\alpha-\mu; z] \, {}_2F_1[-n, 1+\gamma; 1+\gamma-\nu; y] t^n$$

$$= (1-t)^{-\lambda} F_2 \left[\lambda, 1+\alpha, 1+\gamma; 1+\alpha-\mu, 1+\gamma-\nu; \frac{z}{1-t}, \frac{-yt}{1-t}\right], \qquad \text{......(4.6.13)}$$

which is a known result due to Srivastava and Manocha {[94], p.294(1)}.

For $w_2 = 0$, (4.5.3) and (4.5.4) reduces respectively to the following two bilateral generating functions:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_2 \left[-\beta, \lambda+n, -\mu; 1+\alpha-\mu; \frac{w_1}{x-1}, \frac{x}{x-1}\right] {}_2F_1[-n, 1+\gamma; 1+\gamma-\nu; y] t^n$$

$$=(1-t)^{-\lambda} F_k\left[\begin{array}{l} 1+\gamma,-\beta,-\beta,\lambda,-\mu,\lambda;1+\gamma-v;1+\alpha-\mu,-\beta; \\ \qquad \dfrac{-yt}{1-t},\dfrac{x}{x-1},\dfrac{xw_1}{(x-1)(1-t)} \end{array}\right] \qquad \text{......(4.6.14)}$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_3\left[\lambda+n,-\mu,1+\alpha,-\beta;1+\alpha-\mu;xw_1,\frac{x}{x-1}\right]{}_2F_1[-n,1+\gamma;1+\gamma-v;y]t^n$$

$$=(1-t)^{-\lambda} F_N\left[\begin{array}{l} 1+\gamma,-\beta,1+\alpha,\lambda,-\mu,\lambda;1+\gamma-v,1+\alpha-\mu,1+\alpha-\mu; \\ \qquad xw,\dfrac{-yt}{1-t},\dfrac{x}{x-1},\dfrac{xw_1}{(x-1)(1-t)} \end{array}\right]. \qquad \text{......(4.6.15)}$$

CHAPTER-V

Generating Functions of a General Triple Hypergeometric Series $\mathcal{F}^{(3)}[x,y,z]$

CHAPTER-V

GENERATING FUNCTIONS OF A GENERAL TRIPLE HYPERGEOMETRIC SERIES $F^{(3)}$ [x, y, z]

5.1 INTRODUCTION

This chapter deals with a technique of integral operators for obtaining some generating functions involving general triple hypergeometric series $F^{(3)}[x,y,z]$, defined by (1.4.11).

Many generating functions (known and new) involving Appell's double hypergeometric function F_2 and Kampé de Fériet function of two variables $F^{A:B;B'}_{C:D;D'}[x,y]$ defined by (1.3.5) and (1.3.12), respectively are shown here to be special cases of our main results.

The generalized hypergeometric function of two variables, also known as the Kampé de Fériet function, is defined by (1.3.12).

$$F\begin{matrix} A:B;B' \\ C:D;D' \end{matrix}\left[\begin{matrix} (a):(b);(b'); \\ (c):(d);(d'); \end{matrix} x,y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{((a))_{m+n}((b))_m((b'))_n\, x^m\, y^n}{((c))_{m+n}((d))_m((d'))_n\, m!\, n!}, \qquad \ldots\ldots\ldots (5.1.1)$$

where $(a)_n$ is the Pochhammer symbol given by (1.2.6)

and $((a))_m$ will mean the product $\prod_{j=1}^{A}(a_j)_m$.

The general triple hypergeometric series $F^{(3)}[x,y,z]$ is defined by (1.4.11).

$$F^{(3)}[x,y,z]=F^{(3)}\left[\begin{array}{l}(a)::(b);(b');(b''):(c);(c');(c'');\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad x,y,z\\ (e)::(g);(g');(g''):(h);(h');(h'');\end{array}\right]$$

$$=\sum_{m,n,p=0}^{\infty}\frac{((a))_{m+n+p}((b))_{m+n}((b'))_{n+p}((b''))_{p+m}((c))_{m}((c'))_{n}((c''))_{p}\,x^{m}y^{n}z^{p}}{((e))_{m+n+p}((g))_{m+n}((g'))_{n+p}((g''))_{p+m}((h))_{m}((h'))_{n}((h''))_{p}\,m!n!p!}.$$

.... (5.1.2)

Here we mentioned the following standard results, which are used to obtain our main results:

Following results are given in {[69], p.256 (5)}and {[21], p.216 (16)}.

(i) $\int_0^\infty e^{-pt}\,t^{A-1}\,W_{k,\mu}(st)dt$

$$=\frac{s^{-A}\,\Gamma\left(A+\mu+\frac{1}{2}\right)\Gamma\left(A-\mu+\frac{1}{2}\right)}{\Gamma(A-k+1)}\;{}_2F_1\left[\begin{array}{l}A+\mu+\frac{1}{2},A-\mu+\frac{1}{2};\;\frac{s-2p}{2s}\\ \qquad A-k+1\qquad\qquad;\end{array}\right]$$

...... (5.1.3)

$$=\frac{\Gamma\left(A+\mu+\frac{1}{2}\right)\Gamma\left(A-\mu+\frac{1}{2}\right)s^{\mu+\frac{1}{2}}}{\Gamma(A-k+1)\left(p+\frac{s}{2}\right)^{A+\mu+\frac{1}{2}}}\;{}_2F_1\left[\begin{array}{l}A+\mu+\frac{1}{2},\mu-k+\frac{1}{2};\;\frac{p-\frac{s}{2}}{p+\frac{s}{2}}\\ \qquad A-k+1\qquad\qquad;\end{array}\right]$$

...... (5.1.4)

$\mathrm{Re}(A\pm\mu)>-\frac{1}{2}$ *and* $\mathrm{Re}(p+\frac{s}{2})>0.$

where $W_{k,\mu}(x)$ is Whittker function, defined by {[19], p.264 (5)}.

(ii) The following result is given in {[69], p.441 (42)}.

$$ {}_2F_1\left[\begin{matrix}\alpha,\beta; \\ \gamma;\end{matrix} z\right] = {}_4F_3\left[\begin{matrix}\frac{\alpha}{2},\frac{\alpha+1}{2},\frac{\beta}{2},\frac{\beta+1}{2}; \\ \frac{\gamma}{2},\frac{\gamma+1}{2},\frac{1}{2} \quad ;\end{matrix} z^2\right] $$

$$ +\frac{\alpha\beta z}{\gamma}\,{}_4F_3\left[\begin{matrix}\frac{\alpha}{2}+1,\frac{\alpha+1}{2},\frac{\beta}{2}+1,\frac{\beta+1}{2}; \\ \frac{\gamma}{2}+1,\frac{\gamma+1}{2},\frac{3}{2} \quad ;\end{matrix} z^2\right] \qquad \ldots\ldots (5.1.5) $$

(iii)

$$ F_4\left[\alpha,\beta;\gamma,\beta;x,y\right] = (1-x-y)^{-\alpha} $$

$$ H_3\left[\alpha,\gamma-\beta;\gamma;\frac{xy}{(x+y-1)^2},\frac{x}{(x+y-1)}\right] \qquad \ldots\ldots (5.1.6) $$

{[94], p.57(33)}.

(iv)

$$ \int_0^\infty u^{\lambda} e^{-\left(z+\frac{p}{2}\right)u}\, W_{k,\mu}(pu)\, L_m^{(2S_1)}(xu)\, L_n^{(2S_2)}(yu)\, du $$

$$ = \frac{(2S_1+1)_m (2S_2+1)_n \Gamma\left(\lambda+\frac{3}{2}+\mu\right)\Gamma\left(\lambda+\frac{3}{2}-\mu\right) p^{\mu+\frac{1}{2}}}{m!\,n!\,\Gamma(\lambda-k+2)(z+p)^{\lambda+\mu+\frac{3}{2}}} $$

$$ \cdot F^{(3)}\left[\begin{matrix}\lambda+\frac{3}{2}+\mu::\lambda+\frac{3}{2}\ \mu;-;-:-m;-n;\mu-k+\frac{1}{2}; \\ \lambda-k+2::\ -\ ;-;-:2S_1+1;2S_2+1;\ -\ ;\end{matrix} \frac{x}{z+p},\frac{y}{z+p},\frac{z}{z+p}\right] $$

$$ \ldots\ldots (5.1.7) $$

$\mathrm{Re}(\lambda \pm \mu + \frac{3}{2}) > 0; \mathrm{Re}(z+p) > 0.$

{[66], p. 372}.

(v) $$\Psi_2\left[\alpha;\gamma,\gamma;x,-x\right] = {}_2F_3\left[\begin{matrix}\frac{1}{2}\alpha,\frac{1}{2}(\alpha+1) & ; & -x^2\\ \gamma,\ \frac{1}{2}\gamma,\ \frac{1}{2}(\gamma+1); & & \end{matrix}\right] \quad \ldots\ldots (5.1.8)$$

{[93], p. 322 (188)}.

(vi) $F_4\left[a,b;c,c';x,x\right]$

$$= {}_4F_3\left[\begin{matrix}a,b,\frac{1}{2}(c+c'),\frac{1}{2}(c+c'-1); & 4x\\ c,c',c+c'-1 \quad ; & \end{matrix}\right] \quad \ldots\ldots (5.1.9)$$

{Burchnall (9), p. 101} and {cf. [94], p. 55 (16)}.

We first use (5.1.4) to prove the following results, to be used in our investigation later.

$$\int_0^\infty e^{-\left(z+\frac{s}{2}\right)t}\, t^b W_{k,\mu}(st)\, {}_1F_1\left[\begin{matrix}-n\ ; \\ \alpha+1;\end{matrix}\ xt\right] F_1\left[\begin{matrix}-n\ ; \\ \beta+1;\end{matrix}\ yt\right] dt$$

$$= \frac{\Gamma\left(b+\mu+\frac{3}{2}\right)\Gamma\left(b-\mu+\frac{3}{2}\right) s^{\mu+\frac{1}{2}}}{\Gamma(b-k+2)(z+s)^{b+\mu+\frac{3}{2}}}$$

$$\cdot F^{(3)}\left[\begin{matrix}b+\mu+\frac{3}{2} :: b-\mu+\frac{3}{2}\ ;-;-:-n\ ;\ -n\ ;\mu-k+\frac{1}{2}; & \frac{x}{z+s}, \frac{y}{z+s}, \frac{z}{z+s}\\ b-k+2 :: \quad - \quad ;-;-:\alpha+1;\beta+1; \quad - \quad ; & \end{matrix}\right]$$

$$\ldots\ldots (5.1.10)$$

and

$$\int_0^\infty e^{-t}\, t^b W_{k,\mu}(st)\, {}_1F_1\left[\begin{matrix}-n; \\ \alpha;\end{matrix}\ z_1 t\right] {}_1F_1\left[\begin{matrix}\lambda+n; \\ \beta\ \ ;\end{matrix}\ z_2 t\right] dt$$

$$= \frac{\Gamma\left(b+\mu+\frac{3}{2}\right)\Gamma\left(b-\mu+\frac{3}{2}\right) s^{\mu+\frac{1}{2}}}{\Gamma(b-k+2)\left(1+\frac{s}{2}\right)^{b+\mu+\frac{3}{2}}}$$

$$\cdot F^{(3)}\left[\begin{matrix} b+\mu+\frac{3}{2} :: b-\mu+\frac{3}{2} ;-;-:-n;\lambda+n;\mu-k+\frac{1}{2}; \\ b-k+2 :: \quad - \quad ;-;-: \alpha \;;\; \beta \;;\; \quad - \quad ; \end{matrix} \frac{2z_1}{2+s}, \frac{2z_2}{2+s}, \frac{2-s}{2+s}\right]. \qquad \text{...... (5.1.11)}$$

Proof of (5.1.10)

Let us write A for the terms on the left-hand side of (5.1.10). Expanding both ${}_1F_1$ into power series and changing the order of summation and integration, which is permissible due to the absolute convergence of integrals, we have

$$A = \sum_{p,q=0}^{\infty} \frac{(-n)_p (-n)_q \; x^p \, y^q}{(\alpha+1)_p \; (\beta+1)_q \; p!q!} \int_0^{\infty} e^{-\left(z+\frac{s}{2}\right)t} \, t^{b+p+q} \, W_{k,\mu}(st)\,dt \qquad \text{...... (5.1.12)}$$

By using (5.1.4), we get

$$A = \frac{\Gamma\left(b+\mu+\frac{3}{2}\right)\Gamma\left(b-\mu+\frac{3}{2}\right) s^{\mu+\frac{1}{2}}}{\Gamma(b-k+2)(z+s)^{b+\mu+\frac{3}{2}}}$$

$$\cdot \sum_{p,q=0}^{\infty} \frac{\left(b+\mu+\frac{3}{2}\right)_{p+q} \left(b-\mu+\frac{3}{2}\right)_{p+q} (-n)_p (-n)_q \left(\frac{x}{z+s}\right)^p \left(\frac{y}{z+s}\right)^q}{(b-k+2)_{p+q} \; (\alpha+1)_p (\beta+1)_q \; p! \, q!}$$

$$\times {}_2F_1\left[\begin{matrix} b+\mu+\frac{3}{2}+p+q, \mu-k+\frac{1}{2}; \\ b-k+2+p+q \end{matrix} \; \frac{z}{z+s}\right]. \qquad \text{...... (5.1.13)}$$

Now, by expanding ${}_2F_1$ into power series and adjusting the parameters, the result (5.1.13) yields the right-hand side of (5.1.10), and thereby (5.1.10) is proved. The proof of (5.1.11) is the same as above, and, therefore, we omit the details.

5.2 LINEAR GENERATING FUNCTIONS OF $F^{(3)}[x, y, z]$

Following two generating functions for Laguerre polynomials $L_n^{(\alpha)}(x)$ and the confluent hypergeometric function ${}_1F_1$ are found in Srivastava and Manocha {[94], p.134 (16) and p.229 (35)}.

$$\sum_{n=0}^{\infty} \frac{n!}{(\alpha+1)_n} L_n^{(\alpha)}(x) L_n^{(\beta)}(y)\, w^n$$

$$= (1-w)^{-\beta-1} \exp\left(\frac{(x+y)w}{w-1}\right) \phi_3\left[\alpha-\beta; \alpha+1; \frac{xw}{1-w}, \frac{xyw}{(1-w)^2}\right], \quad |w| < 1.$$

$$\ldots\ldots (5.2.1)$$

where ϕ_3 is Humbert's function defined by (1.3.8)

and

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, {}_1F_1\left[\begin{matrix} -n; \\ \alpha\ ; \end{matrix} z_1\right] {}_1F_1\left[\begin{matrix} \lambda+n; \\ \beta\ ; \end{matrix} z_2\right] w^n$$

$$= (1-w)^{-\lambda}\, F^{1:0;0}_{0:1;1}\left[\begin{matrix} \lambda : -; -; \\ - : \alpha ; \beta ; \end{matrix} \frac{z_1 w}{w-1}, \frac{z_2}{1-w}\right], \qquad |w| < 1 \qquad \ldots\ldots (5.2.2)$$

Now, in terms of confluent hypergeometric function ${}_1F_1$, we can rewrite (5.2.1) as:

$$\sum_{n=0}^{\infty}\frac{(\beta+1)_n}{n!}\ {}_1F_1\left[\begin{matrix}-n\ ;\\ \alpha+1;\end{matrix}x\right]\ {}_1F_1\left[\begin{matrix}-n\ ;\\ \beta+1;\end{matrix}y\right]w^n$$

$$=(1-w)^{-\beta-1}\exp\left(\frac{(x+y)\,w}{w-1}\right)\phi_3\left[\alpha-\beta;\alpha+1;\frac{xw}{1-w},\frac{xyw}{(1-w)^2}\right],\quad |w|<1$$

...... (5.2.3)

In (5.2.3), if we replace x by xt and y by yt, multiply both the sides by $e^{-\left(z+\frac{s}{2}\right)t}\,t^b\,W_{k,\mu}(st)$ and integrate with respect to t between the limits 0 to ∞ by using the results (5.1.3) and (5.1.10) and adjusting the parameters and by putting $b+\mu+\dfrac{3}{2}=A,\quad b-\mu+\dfrac{3}{2}=B,\quad b-k+2=C$ and s = 1, we get

$$\sum_{n=0}^{\infty}\frac{(\beta+1)_n\,w^n}{n!}\ F^{(3)}\left[\begin{matrix}A::B\ ;-;-:-n;-n;C-B;\\ B::-\ ;-;-:\alpha+1;\beta+1;\ -;\end{matrix}\frac{x}{z+1};\frac{y}{z+1};\frac{z}{z+1}\right]$$

$$=(Z+1)^A\,(1-w)^{-\beta-1}\left\{\sum_{q,r=0}^{\infty}\frac{(\alpha-\beta)_q\left(\dfrac{xw}{1-w}\right)^q\left(\dfrac{xyw}{(1-w)^2}\right)^r}{(\alpha+1)_{q+r}\ q!\ r!}\right\}\frac{(A)_{q+2r}\,(B)_{q+2r}}{(C)_{q+2r}}$$

$${}_2F_1\left[\begin{matrix}A+q+2r,\ B+q+2r;\\ C+q+2r;\end{matrix}\ \frac{(x+y)w}{w-1}\ -z\right].\qquad ......\ (5.2.4)$$

We adopt the same analysis that is employed to obtain (5.2.4) and, we use (5.1.4) and (5.1.11) to (5.2.2), following result is obtained

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n w^n}{n!} F^{(3)}\left[\begin{matrix} b+\mu+\frac{3}{2} :: b-\mu+\frac{3}{2} ; - ; - : -n ; \lambda+n ; \mu-k+\frac{1}{2} ; \\ b-k+2 :: \quad - \quad ; - ; - : \alpha \;;\; \beta \;;\; - \;; \end{matrix} \frac{2z_1}{2+s}, \frac{2z_2}{2+s}, \frac{2-s}{2+s}\right]$$

$$= (1-w)^{-\lambda} F^{(3)}\left[\begin{matrix} b+\mu+\frac{3}{2} :: b-\mu+\frac{3}{2}, \lambda ; - ; - : - ; - ; \mu-k+\frac{1}{2} ; \\ b-k+2 :: \quad - \quad ; - ; - : \alpha ; \beta ; \quad - \quad ; \end{matrix} \frac{2z_1 w}{(w-1)(2+s)}, \frac{2z_2}{(1-w)(2+s)}, \frac{2-s}{2+s}\right] \qquad \ldots\ldots (5.2.5)$$

Replacing $b+\mu+\frac{3}{2}, b-\mu+\frac{3}{2}, b-k+2, z_1, z_2$ *and* s by *a, b, c,* 2*x*, 2*y* and 2*z*, respectively, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n w^n}{n!} F^{(3)}\left[\begin{matrix} a :: b \; ; - ; - : -n ; \lambda+n ; c-b ; \\ c :: - \; ; - ; - : \alpha \;;\; \beta \;;\; - \;; \end{matrix} \frac{2x}{1+z}, \frac{2y}{1+z}, \frac{1-z}{1+z}\right]$$

$$= (1-w)^{-\lambda} F^{(3)}\left[\begin{matrix} a :: b, \lambda ; - ; - : - ; - ; c-b ; \\ c :: \quad - \quad ; - ; - : \alpha ; \beta ; \; - \;; \end{matrix} \frac{2xw}{(w-1)(1+z)}, \frac{2y}{(1-w)(1+z)}, \frac{1-z}{1+z}\right].$$

$$\ldots\ldots (5.2.6)$$

5.3 SPECIAL CASES

In this section we will mention some special cases of our results (5.2.4) and (5.2.6).

(i) Letting $z = 0$ in (5.2.4) and taking $B = C$, we get

$$\sum_{n=0}^{\infty}\frac{(\beta+1)_n w^n}{n!} F_2[A,-n,-n;\alpha+1,\beta+1;x,y]$$

$$=(1-w)^{A-\beta-1}(1-w+xw+yw)^{-A}$$

$$\times H_3\left[A,\alpha-\beta;\alpha+1;\frac{xyw}{(1-w+xw+yw)^2},\frac{xw}{(1-w+xw+yw)}\right] \quad \ldots\ldots (5.3.1)$$

which is a known result {[82], p.681 (2.2)}.

where F_2 is Appell's Function defined by (1.3.5) and H_3 is Horn's Function defined by (1.3.10).

(ii) Putting $\alpha=\beta$ in (5.2.4) and using (5.1.5), we get

$$\sum_{n=0}^{\infty}\frac{(\alpha+1)_n w^n}{n!} F^{(3)}\left[\begin{matrix}A::B;-;-:\ -n\ ;\ -n\ ;C-B;\\ C::-;-;-:\alpha+1;\alpha+1;\ \ -\ \ ;\end{matrix}\ \frac{x}{z+1},\frac{y}{z+1},\frac{z}{z+1}\right]$$

$$=(z+1)^A(1-w)^{-\alpha-1}$$

$$\left\{F^{4:0;0}_{2:1;1}\left[\begin{matrix}\frac{A}{2},\frac{A+1}{2},\frac{B}{2},\frac{B+1}{2}: & - & ; & -; \\ \frac{C}{2},\frac{C+1}{2}: & \alpha+1 & ; & \frac{1}{2};\end{matrix}\ \frac{4xyw}{(1-w)^2},\left(\frac{(x+y)w}{w-1}-z\right)^2\right]\right.$$

$$+\frac{AB}{C}\left[\frac{(x+y)w}{w-1}-z\right]$$

$$\left.\times F^{4:0;0}_{2:1;1}\left[\begin{matrix}\frac{A}{2}+1,\frac{A+1}{2},\frac{B}{2}+1,\frac{B+1}{2} & : & - & ; & - & ; \\ \frac{C}{2}+1,\frac{C+1}{2} & : & \alpha+1 & ; & \frac{3}{2} & ;\end{matrix}\ \frac{4xyw}{(1-w)^2},\left(\frac{(x+y)w}{w-1}-z\right)^2\right]\right\}$$

$$\ldots\ldots\ldots (5.3.2)$$

(iii) Taking z = 0 in (5.3.2) and replacing y by $-x$, we get

$$\sum_{n=0}^{\infty} \frac{(\alpha+1)_n w^n}{n!} F^{2:1;1}_{1:1;1}\left[\begin{matrix} A, B & : & -n\ ; & -n\ ; \\ C & : & \alpha+1; & \alpha+1; \end{matrix}\ x,\ -x\right]$$

$$= (1-w)^{-\alpha-1}\ {}_4F_3\left[\begin{matrix} \frac{A}{2}, \frac{A+1}{2}, \frac{B}{2}, \frac{B+1}{2} & ; \\ \frac{C}{2}, \frac{C+1}{2}, \ \alpha+1 & ; \end{matrix}\ \frac{-4x^2 w}{(1-w)^2}\right] \qquad \ldots\ldots (5.3.3)$$

where ${}_pF_q$ is generalized hypergeometric function defined by (1.2.23).

(iv) On letting $z = 0$, $B = C$, (5.3.2) reduces to

$$\sum_{n=0}^{\infty} \frac{(\alpha+1)_n w^n}{n!} F_2\left[A, -n, -n; \alpha+1, \alpha+1; x, y\right]$$

$$= (1-w)^{-\alpha-1}\left\{F_4\left[\frac{A}{2}, \frac{A+1}{2}; \alpha+1, \frac{1}{2}; \frac{4xyw}{(1-w)^2}, \left(\frac{(x+y)w}{w-1}\right)^2\right]\right.$$

$$\left. + \frac{A(x+y)w}{w-1}\ F_4\left[\frac{A}{2}+1,\ \frac{A+1}{2};\ \alpha+1, \frac{3}{2}; \frac{4xyw}{(1-w)^2}, \left(\frac{(x+y)w}{w-1}\right)^2\right]\right\}$$

$$\ldots\ldots (5.3.4)$$

where F_4 is Appell's Function defined by (1.3.7).

(v) Further replacing y by $-x$ in (5.3.4), we get

$$\sum_{n=0}^{\infty} \frac{(\alpha+1)_n w^n}{n!} F_2\left[A, -n, -n; \alpha+1, \alpha+1; x, -x\right]$$

$$= (1-w)^{-\alpha-1}\ {}_2F_1\left[\begin{matrix} \frac{A}{2}, \ \frac{A+1}{2} & ; \\ \alpha+1 & ; \end{matrix}\ \frac{-4x^2 w}{(1-w)^2}\right]. \qquad \ldots\ldots (5.3.5)$$

(vi) Taking $z = 1$ in (5.2.6), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n w^n}{n!} F^{2:1;1}_{1:1;1}\left[\begin{matrix} a, b : -n ; \lambda + n ; \\ c : \alpha ; \beta ; \end{matrix} x, y\right]$$

$$= (1-w)^{-\lambda} F^{3:0;0}_{1:1;1}\left[\begin{matrix} a, b, \lambda : - ; - ; \\ c : \alpha ; \beta ; \end{matrix} \frac{xw}{w-1}, \frac{y}{1-w}\right]. \qquad \text{......... (5.3.6)}$$

(vii) Further taking $b = c$, $\beta = \lambda$ in (5.3.6) and using the result (5.1.6), we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n w^n}{n!} F_2[a, -n, \lambda + n; \alpha, \lambda; x, y]$$

$$= (1-w)^{a-\lambda} (1 - w + xw - y)^{-a}$$

$$H_3\left[a, \alpha - \lambda; \alpha; \frac{-xyw}{(w - xw + y - 1)^2}, \frac{-xw}{w - xw + y - 1}\right] \qquad \text{...... (5.3.7)}$$

which for $\alpha = \lambda$ reduces to

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n w^n}{n!} F_2[a, -n, \lambda + n; \lambda, \lambda; x, y]$$

$$= (1-w)^{a-\lambda} \theta^{-a} {}_2F_1\left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2} ; \\ \lambda ; \end{matrix} \frac{-4xyw}{\theta^2}\right], \qquad \text{........ (5.3.8)}$$

where $\theta = (1 - w + xw - y)$.

5.4 DOUBLE GENERATING FUNCTIONS OF $F^{(3)}(x, y, z)$

Consider the two generating functions for the product of a pair of Laguerre polynomials $L_m^{(\alpha)}(x)$ Exton {[26], p. 406, 404}.

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\, x^m (-x)^n}{((g))_{m+n}\, (h)_m\, (h)_n} L_m^{(h-1)}(y) L_n^{(h-1)}(y)$$

$$= {}_{2D}F_{2G+3}\left[\begin{matrix} \left(\frac{d}{2}\right), \left(\frac{d}{2}+\frac{1}{2}\right) & ; \\ \left(\frac{g}{2}\right), \left(\frac{g}{2}+\frac{1}{2}\right), h, \frac{h}{2}, \frac{h}{2}+\frac{1}{2}; & \end{matrix} -4^{D-G-1} x^2 y^2\right] \qquad \ldots\ldots (5.4.1)$$

and

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\, x^m (-x)^n}{((g))_{m+n}\, (h)_m\, (h')_n} L_m^{(h-1)}(y) L_n^{(h'-1)}(-y)$$

$$= {}_{D+2}F_{G+3}\left[\begin{matrix} (d), \frac{h+h'-1}{2}, \frac{h+h'}{2}; \\ (g), h, h', h+h'-1 \quad ; \end{matrix} 4xy\right]. \qquad \ldots\ldots\ldots (5.4.2)$$

In (5.4.1), if we replace y by yu, multiply both sides by $u^{\lambda} e^{-\left(z+\frac{p}{2}\right)u} W_{k,\mu}(pu)$ and then take the integration with respect to u between the limits 0 and ∞ with the help of the results (5.1.3) and (5.1.7) and adjusting the parameters, we obtain the following generating function:

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\, x^m (-x)^n}{((g))_{m+n}\, m!n!} F^{(3)}\left[\begin{matrix} a :: b; -; - : -m; -n; c-b; \\ c :: -; -; - : h; h; \quad - \end{matrix} ; \frac{y}{z+1}, \frac{y}{z+1}, \frac{z}{z+1}\right]$$

$$=(z+1)^a \sum_{r=0}^{\infty} \frac{\left(\left(\frac{d}{2}\right)\right)_r \left(\left(\frac{d}{2}+\frac{1}{2}\right)\right)_r \left(-4^{D-G-1} x^2 y^2\right)^r}{\left(\left(\frac{g}{2}\right)\right)_r \left(\left(\frac{g}{2}+\frac{1}{2}\right)\right)_r (h)_r \left(\frac{h}{2}\right)_r \left(\frac{h}{2}+\frac{1}{2}\right)_r r!} \frac{(a)_{2r}(b)_{2r}}{(c)_{2r}}$$

$${}_2F_1\left[\begin{matrix} a+2r, b+2r; \\ c+2r \quad ; \end{matrix} -z\right] \qquad \ldots\ldots\ldots (5.4.3)$$

where $a = \lambda+\frac{3}{2}+\mu$, $b = \lambda+\frac{3}{2}-\mu$ and $c = \lambda-k+2$.

The above result on letting z → 0, yields

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\, x^m (-x)^n}{((g))_{m+n}\, m!n!} F^{2:1;1}_{1:1;1}\left[\begin{matrix} a,b :-m;-n; \\ c :h;h \quad ; \end{matrix} y,y\right]$$

$$= {}_{2D+4}F_{2G+4}\left[\begin{matrix} \left(\frac{d}{2}\right),\left(\frac{d}{2}+\frac{1}{2}\right),\frac{a}{2},\frac{a}{2}+\frac{1}{2},\frac{b}{2},\frac{b}{2}+\frac{1}{2} \quad ; \\ \left(\frac{g}{2}\right),\left(\frac{g}{2}+\frac{1}{2}\right),\frac{c}{2},\frac{c}{2}+\frac{1}{2},h,\frac{h}{2},\frac{h}{2}+\frac{1}{2}; \end{matrix} -4^{D-G-1}\cdot 4\, x^2 y^2\right]$$

$$\ldots\ldots\ldots (5.4.4)$$

We adopt the same analysis that is employed to obtain (5.4.3) and, we use (5.1.7) and (5.1.4) to (5.4.2), following result is obtained

$$\sum_{m,n=0}^{\infty} \frac{((d))_{m+n}\, x^m (-x)^n}{((g))_{m+n}\, m!n!} F^{(3)}\left[\begin{matrix} a :: b;-;-:-m;-n;c-b; \\ c :: -;-;-:h\ ;h'; \quad - \ ; \end{matrix} \frac{y}{z+1},\frac{-y}{z+1},\frac{z}{z+1}\right]$$

$$=\sum_{r=0}^{\infty} \frac{((d))_r \left(\frac{h+h'-1}{2}\right)_r \left(\frac{h+h'}{2}\right)_r \left(\frac{4xy}{z+1}\right)^r}{((g))_r (h)_r (h')_r (h+h'-1)_r\, r!} \frac{(a)_r (b)_r}{(c)_r}\, {}_2F_1\left[\begin{matrix} a+r, c-b; \\ c+r \quad ; \end{matrix} \frac{z}{z+1}\right]$$

$$\ldots\ldots\ldots (5.4.5)$$

The above result on letting z → 0 yields

$$\sum_{m,n=0}^{\infty}\frac{((d))_{m+n}\,x^{m}(-x)^{n}}{((g))_{m+n}\,m!n!}F^{2:1;1}_{1:1;1}\left[\begin{matrix}a,b:-m;-n;\\ c:h,h'\quad ;\end{matrix}\,y,-y\right]$$

$$={}_{D+4}F_{G+4}\left[\begin{matrix}(d),a,b,\dfrac{h+h'-1}{2},\dfrac{h+h'}{2};4xy\\ (g),c,h,h',h+h'-1\quad ;\end{matrix}\right]. \qquad \text{.........} (5.4.6)$$

5.5 SPECIAL CASES

In this section we will mention some special cases of our results (5.4.3), (5.4.4), (5.4.5) and (5.4.6).

(i) On putting $D = 1$, $G = 0$ and $d = h = \alpha + 1$ in (5.4.3) and using (5.1.5), we get

$$\sum_{m,n=0}^{\infty}\frac{(\alpha+1)_{m+n}\,x^{m}(-x)^{n}}{m!n!}F^{(3)}\left[\begin{matrix}a::b;-;-:-m;-n;c-b;\\ c::-;-;-:\alpha+1;\alpha+1;\;-\;;\end{matrix}\frac{y}{z+1},\frac{y}{z+1},\frac{z}{z+1}\right]$$

$$=(z+1)^{a}\left\{F^{4:0;0}_{2:1;1}\left[\begin{matrix}\dfrac{a}{2},\dfrac{a+1}{2},\dfrac{b}{2},\dfrac{b+1}{2}:\quad -;-;\\ \dfrac{c}{2},\dfrac{c+1}{2}:\alpha+1;\dfrac{1}{2};\end{matrix}\;-4x^{2}y^{2},z^{2}\right]\right.$$

$$\left.-\frac{abz}{c}F^{4:0;0}_{2:1;1}\left[\begin{matrix}\dfrac{a}{2}+1,\dfrac{a+1}{2},\dfrac{b}{2}+1,\dfrac{b+1}{2}:-;-;\\ \dfrac{c}{2}+1,\dfrac{c+1}{2}\quad :\alpha+1;\dfrac{3}{2};\end{matrix}\;-4x^{2}y^{2},z^{2}\right]\right\} \qquad \text{.........} (5.5.1)$$

which is a known result due to Pathan and Khan {[68], p. 130 (3.5)}.

(ii) If in (5.4.3), we put $D = 1$, $G = 0$, $d = h = 1 + \alpha$ and $b = c$ and use (5.1.5), we get

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n} x^m (-x)^n}{m!n!} F_2\left[a,-m,-n;\alpha+1,\alpha+1;\frac{y}{z+1},\frac{y}{z+1}\right]$$

$$= {}_4F_3\left[\begin{array}{l} \frac{a}{4},\frac{a+2}{4},\frac{a+1}{4},\frac{a+3}{4}; \\ \frac{\alpha+1}{2},\frac{\alpha+2}{2},\frac{1}{2} \end{array} ; \left(\frac{-4x^2y^2}{(z+1)^2}\right)^2\right]$$

$$-\frac{a(a+1)}{\alpha+1}\left(\frac{xy}{z+1}\right)^2 {}_4F_3\left[\begin{array}{l} \frac{a+4}{4},\frac{a+2}{4},\frac{a+5}{4},\frac{a+3}{4}; \\ \frac{\alpha+3}{2};\frac{\alpha+2}{2},\frac{3}{2} \end{array} ; \left(\frac{-4x^2y^2}{(z+1)^2}\right)^2\right], \qquad \text{...... (5.5.2)}$$

where F_2 is Appell's Function defined by (1.3.5).

(iii) On putting $D = G = 0$ and $b = c$ and using (5.1.8) equation (5.4.4) reduces to

$$\sum_{m,n=0}^{\infty} \frac{x^m (-x)^n}{m!n!} F_2[a,-m,-n;h,h;y,y]$$

$$=\psi_2[a;h,h;xy,-xy], \qquad \text{......... (5.5.3)}$$

where ψ_2 is the confluent hypergeometric function of two variables defined by (1.3.9).

(iv) Further, if in (5.4.4), we put $D = 1$, $G = 0$ and $d = h = \alpha + 1$, then it reduces to another known result due Pathan and Khan {[68], p.130 (3.7)}.

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_n\, x^m (-x)^n}{m!n!} F^{2:1;1}_{1:1;1}\left[\begin{matrix} a,b : -m \,;\, -n \,; \\ c \;:\alpha+1;\alpha+1; \end{matrix} y, y\right]$$

$$= {}_4F_3\left[\begin{matrix} a/2, a/2+1/2, b/2, b/2+1/2 \,; \\ c/2, c/2+1/2, \alpha+1 \quad ; \end{matrix} -4x^2y^2\right]. \qquad \ldots\ldots\ldots (5.5.4)$$

(v) On putting $a = c$, $D = 1$, $G = 0$ and $d = h' = h$ in (5.4.5), we get

$$\sum_{m,n=0}^{\infty} \frac{(h)_{m+n}\, x^m (-x)^n}{m!n!} F^{(3)}\left[\begin{matrix} - :: b;-;- : -m;-n;c-b; \\ - :: -;-;- : \;h;\;\; h;\;\; - \;\; ; \end{matrix} \frac{y}{z+1}, \frac{-y}{z+1}, \frac{z}{z+1}\right]$$

$$= (z+1)^{c-b}\, {}_2F_1\left[\begin{matrix} h-\frac{1}{2}, b; \\ 2h-1 \quad ; \end{matrix} \frac{4xy}{z+1}\right]. \qquad \ldots\ldots\ldots (5.5.5)$$

(vi) If in (5.3.6), we put $D = 1$, $G = 0$, $b = c$, and $d = h = h' = \alpha + 1$ and using the results (5.1.5) and (5.1.9), we get

$$\sum_{m,n=0}^{\infty} \frac{(\alpha+1)_{m+n}\, x^m (-x)^n}{m!n!} F_2\left[a, -m, -n; \alpha+1, \alpha+1; y, -y\right]$$

$$= F_4\left[\frac{a}{2}, \frac{a+1}{2}; \alpha+1, \frac{1}{2}; 4x^2y^2, 4x^2y^2\right]$$

$$+ 2axy F_4\left[\frac{a+2}{2}, \frac{a+1}{2}; \alpha+1, \frac{3}{2}; 4x^2y^2, 4x^2y^2\right], \qquad \ldots\ldots\ldots (5.5.6)$$

where F_4 is Appell's Function defined by (1.3.7).

CHAPTER-VI

On Generating Functions Involving Laguerre, Legendre and Gegenbauer Polynomials

CHAPTER-VI

ON GENERATING FUNCTIONS INVOLVING LAGUERRE, LEGENDRE AND GEGENBAUER POLYNOMIALS

6.1 INTRODUCTION

The group theoretic method for obtaining generating functions have received much attention. L. Weisner [97] made a significant study in this direction. He obtained generating functions for hypergeometric function by group theoretic method. W. Miller [50] and McBride [47] present Weisner's method in a systematic manner and thereby lay its firm foundation.

The main object of the present chapter is to obtain a general class of generating functions for Laguerre, Legendre and Gegenbauer polynomials with the help of Weisner's group theoretic method.

Actually, we have extended the results given by Majumdar [45] and Kar and Basu [32] to new general theorems on general class of generating functions.

The Legendre polynomial $P_n(x)$ is defined as {[71] p.166 (3)}

$$P_n(x) = (-1)^n \, {}_2F_1\left[-n, n+1; 1; \frac{1+x}{2}\right] \qquad \text{........ (6.1.1)}$$

The Laguerre polynomial $L_n^{(\alpha)}(x)$ is defined as {[71] p.200 (1)}

$$L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n!}\,{}_1F_1\left[-n;1+\alpha;x\right] \qquad \text{........ (6.1.2)}$$

And, the Gegenbauer polynomial $C_n^{v}(x)$ is defined as {[71] p.279 (15)}

$$C_n^{v}(x)=\frac{(2v)_n}{n!}\,{}_2F_1\left[-n,2v+n;v+\frac{1}{2};\frac{1-x}{2}\right]. \qquad \text{........ (6.1.3)}$$

The main results of this chapter is given below:

6.2 MAIN RESULTS

Theorem-1: If there exists the following generating function

$$G(x,w,u,j)=\sum_{n=0}^{\infty}a_n\,L_n^{(\alpha)}(x)L_m^{(n)}(w)C_s^{n}(u)\,j^n \qquad \text{........ (6.2.1)}$$

then

$$(1+j)^{\alpha}\exp(-jx-j)\,(1-2j)^{-s/2}\,G\left[x(1+j),w+j,\frac{u}{\sqrt{1-2j}},\frac{jz}{1-2j}\right]$$

$$=\sum_{n,p,q,r=0}^{\infty}\frac{a_n\,(n+1)_p\,(-1)^q\,2^r\,(n)_r\,j^{n+p+q+r}}{p!\,q!\,r!}\,L_{n+p}^{(\alpha-p)}(x)\,L_m^{(n+q)}(w)\,C_s^{n+r}(u)z^n$$

$$\text{..... (6.2.2)}$$

Theorem-2: If there exists the following generating function

$$G(x,u,z,w)=\sum_{n=0}^{\infty}a_n w^n\,P_n(x)C_s^{n}(u)\,L_m^{(n)}(z) \qquad \text{........ (6.2.3)}$$

then the following general class of generating relation holds true:

$$(1+2wx+w^2)^{-1/2}(1-2w)^{-s/2}e^{-w}$$

$$G\left[\frac{x+w}{\sqrt{1+2wx+w^2}},\frac{u}{\sqrt{1-2w}},z+w,\frac{wj}{(1-2w)\sqrt{1+2wx+w^2}}\right]$$

$$=\sum_{n,p,q,r=0}^{\infty}\frac{a_n(-1)^{p+r}(n+1)_p\,2^q\,(n)_q\,w^{n+p+q+r}}{p!\,q!\,r!}P_{n+p}(x)C_s^{n+q}(u)L_m^{n+r}(z)\,j^n$$

..... (6.2.4)

Proof of Theorem (1):

For the Laguerre polynomials $L_n^{(\alpha)}(x)$ and $L_m^{(n)}(w)$ and for the Gegenbauer polynomials $C_s^n(u)$, we consider the following three operators R_1, R_2 {cf. [45] and [17]} and R_3 {cf. [32]}:

$$R_1 = xy^{-1}\, z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - xy^{-1}z \qquad \text{........ (6.2.5)}$$

$$R_2 = t\frac{\partial}{\partial w} - t \qquad \text{........ (6.2.6)}$$

and

$$R_3 = uv\frac{\partial}{\partial u} + 2v^2\frac{\partial}{\partial v} + sv \qquad \text{........ (6.2.7)}$$

such that

$$R_1\left(L_n^{(\alpha)}(x)\,y^{\alpha}\,z^n\right) = (n+1)L_{n+1}^{(\alpha-1)}(x)\;y^{\alpha-1}\;z^{n+1} \qquad \text{........ (6.2.8)}$$

$$R_2\left(L_m^{(n)}(w)t^n\right) = -L_m^{(n+1)}(w)\;t^{n+1} \qquad \text{........ (6.2.9)}$$

and

$$R_3\ \left(C_s^n(u)v^n\right) = 2nC_s^{n+1}(u)\ v^{n+1} \qquad \ldots\ldots\ldots (6.2.10)$$

and also

$$e^{jR_1}\ f\ (x,y,z) = \exp(-j\,x\,y^{-1}z)\,f\left(x(1+jy^{-1}z), y+jz, z\right) \qquad \ldots\ldots\ldots (6.2.11)$$

$$e^{jR_2}\ f\ (w,t)\ = \exp(-jt)\,f\left(w+jt,t\right) \qquad \ldots\ldots\ldots (6.2.12)$$

$$e^{jR_3}\ f\ (u,v) = (1-2jv)^{-s/2}\ f\left(\frac{u}{\sqrt{1-2jv}}, \frac{v}{1-2jv}\right). \qquad \ldots\ldots\ldots (6.2.13)$$

Let us now consider the formula

$$G(x,w,u,j) = \sum_{n=0}^{\infty} a_n\, L_n^{(\alpha)}\ (x) L_m^{(n)}(w) C_s^n(u)\, j^n \qquad \ldots\ldots\ldots (6.2.14)$$

Replacing j by $j\ z\ t\ v$ and then multiplying both sides of (6.2.14) by y^α and operating both sides by

$e^{jR_1}\ e^{jR_2}\ e^{jR_3}$, we get

$$e^{jR_1}\ e^{jR_2}\ e^{jR_3}\left(y^\alpha\, G(x,w,u,jztv)\right)$$

$$= e^{jR_1}\ e^{jR_2}\ e^{jR_3}\left(\sum_{n=0}^{\infty} a_n\, L_n^{(\alpha)}\ (x)\, y^\alpha\, z^n L_m^{(n)}(w) t^n\, C_s^n(u) v^n\, j^n\right) \qquad \ldots\ldots\ldots (6.2.15)$$

The left hand side of (6.2.15) becomes

$$y^\alpha\, (1+j\frac{z}{y})^\alpha\ \exp(-jxy^{-1}\,z - j\,t)\,(1-2\,j\,v)^{-s/2}$$

$$\times G\left(x(1+j\,y^{-1}\,z), w+j\,t, \frac{u}{\sqrt{1-2\,jv}}, \frac{j\,v\,t\,z}{1-2\,jv}\right) \qquad \ldots\ldots\ldots (6.2.16)$$

and the right hand side of (6.2.15) becomes

$$y^{\alpha} \sum_{n,p,q,r=0}^{\infty} \frac{a_n (n+1)_p (-1)^q \, 2^r \, (n)_r \, j^{n+p+q+r}}{p! \, q! \, r!} \left(\frac{z}{y}\right)^p z^n \, t^{n+q} \, v^{n+r}$$

$$L_{n+p}^{(\alpha-p)}(x) \, L_m^{(n+q)}(w) \, C_s^{n+r}(u) \qquad \ldots\ldots\ (6.2.17)$$

Now equating both (6.2.16) and (6.2.17) and putting $\frac{z}{y}=1$, $v = t = 1$, then we obtain (6.2.2), which is Theorem (1).

Proof of Theorem (2): For Legendre polynomials $P_n(x)$ and Gegenbauer polynomials $C_s^{(n)}(u)$ and Laguerre polynomials $L_m^{(n)}(z)$, we consider the following three linear partial differential operators A_1, A_2 {cf. [32]} and A_3 {cf. [45]}:

$$A_1 = (1-x^2)\, y \frac{\partial}{\partial x} - x y^2 \frac{\partial}{\partial y} - x y \qquad \ldots\ldots\ (6.2.18)$$

$$A_2 = uv \frac{\partial}{\partial u} + 2v^2 \frac{\partial}{\partial v} + sv \qquad \ldots\ldots\ (6.2.19)$$

$$A_3 = t \frac{\partial}{\partial z} - t \qquad \ldots\ldots\ (6.2.20)$$

such that

$$A_1 \left[P_n(x)\, y^n\right] = -(n+1)\, P_{n+1}(x)\, y^{n+1} \qquad \ldots\ldots\ (6.2.21)$$

$$A_2 \left[C_s^n(u)\, v^n\right] = 2n \; C_s^{n+1}(u)\, v^{n+1} \qquad \ldots\ldots\ (6.2.22)$$

$$A_3 \left[L_m^{(n)}(z)\, t^n\right] = -L_m^{(n+1)}(z)\, t^{n+1} \qquad \ldots\ldots\ (6.2.23)$$

and also

$$e^{wA_1} f(x,y) = (1+2wxy+w^2y^2)^{-1/2} f\left(\frac{x+wy}{\sqrt{1+2wxy+w^2y^2}}, \frac{y}{\sqrt{1+2wxy+w^2y^2}}\right) \qquad \text{........ (6.2.24)}$$

$$e^{wA_2} f(u,v) = (1-2wv)^{-s/2} f\left(\frac{u}{\sqrt{1-2wv}}, \frac{v}{1-2wv}\right) \qquad \text{........ (6.2.25)}$$

$$e^{wA_3} f(z,t) = e^{-wt} f(z+wt,t) \qquad \text{........ (6.2.26)}$$

Let us now consider the formula

$$G(x,u,z,w) = \sum_{n=0}^{\infty} a_n w^n P_n(x) C_s^n(u) L_m^{(n)}(z) \qquad \text{........ (6.2.27)}$$

Replacing w by $wytvj$

$$G(x,u,z,wytvj) = \sum_{n=0}^{\infty} a_n (wj)^n P_n(x) y^n C_s^n(u) v^n L_m^{(n)}(z) t^n \qquad \text{........ (6.2.28)}$$

Operating $e^{wA_1} e^{wA_2} e^{wA_3}$ on both sides of (2.28), we get

$$e^{wA_1} e^{wA_2} e^{wA_3} G[(x,u,z,wytvj)]$$

$$= e^{wA_1} e^{wA_2} e^{wA_3} \left\{ \sum_{n=0}^{\infty} a_n (wj)^n P_n(x) y^n C_s^n(u) v^n L_m^{(n)}(z) t^n \right\} \qquad \text{........ (6.2.29)}$$

The left member of (6.2.29) becomes

$$\left(1+2wxy+w^2y^2\right)^{-1/2} (1-2wv)^{-s/2} . e^{-wt}$$

$$\times G\left[\frac{x+wy}{\sqrt{1+2wxy+w^2y^2}}, \frac{u}{\sqrt{1-2wv}}, z+wt, \frac{wytvj}{(1-2wv)\sqrt{1+2wxy+w^2y^2}}\right]$$

.... (6.2.30)

Also the right member of (6.2.29) becomes

$$\sum_{n,p,q,r=0}^{\infty} \frac{a_n (-1)^{p+r} (n+1)_p 2^q (n)_q w^{n+p+q+r}}{p!\, q!\, r!} P_{n+p}(x) C_s^{n+q}(u) L_m^{n+r}(z) j^n$$

.... (6.2.31)

Equating (6.2.30) and (6.2.31) and putting $y = t = v = 1$, we have

$$(1+2wx+w^2)^{-1/2} (1-2w)^{-s/2} e^{-w}$$

$$G\left[\frac{x+w}{\sqrt{1+2wx+w^2}}, \frac{u}{\sqrt{1-2w}}, z+w, \frac{wj}{(1-2w)\sqrt{1+2wx+w^2}}\right]$$

$$= \sum_{n,p,q,r=0}^{\infty} \frac{a_n (-1)^{p+r} (n+1)_p 2^q (n)_q w^{n+p+q+r}}{p!\, q!\, r!} P_{n+p}(x) C_s^{n+q}(u) L_m^{n+r}(z) j^n$$

.... (6.2.32)

This completes the proof of Theorem-2.

3. APPLICATIONS

(I) If we put $s = 0$, and use

$$\sum_{r=0}^{\infty} \frac{(n)_r}{r!} (2j)^r = (1-2j)^{-n} ,$$

our Theorem (1) becomes

If

$$G(x,w,j) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x) L_m^{(n)}(w) j^n \qquad \text{........ (6.3.1)}$$

then

$$(1+j)^{\alpha} \exp(-jx-j)\ G\left[x(1+j), w+j, \frac{jz}{1-2j}\right]$$

$$= \sum_{n,p,q,=0}^{\infty} \frac{a_n (n+1)_p (-1)^q j^{n+p+q}}{p!\, q!} L_{n+p}^{(\alpha-p)}(x)\, L_m^{(n+q)}(w) z^n (1-2j)^{-n} \quad \ldots\ldots\ldots (6.3.2)$$

Replacing $\frac{z}{1-2j}$ by z in (6.3.2), we get

$$(1+j)^{\alpha} \exp(-jx-j)\ G[x(1+j), w+j, jz]$$

$$= \sum_{n,p,q,=0}^{\infty} \frac{a_n (n+1)_p (-1)^q j^{n+p+q}}{p!\, q!} L_{n+p}^{(\alpha-p)}(x)\, L_m^{(n+q)}(w) z^n \quad \ldots\ldots\ldots (6.3.3)$$

This was derived by Majumdar {[45], p. 196}.

(II) In Theorem (1) we set m = 0 and use

$$\sum_{q=0}^{\infty} \frac{(-j)^q}{q!} = e^{-j} ,$$

we obtain

$$(1+j)^{\alpha} \exp(-jx-j)\ (1-2j)^{-s/2}\ G\left[x(1+j), \frac{u}{\sqrt{1-2j}}, \frac{jz}{1-2j}\right]$$

$$= e^{-j} \sum_{n,p,r=0}^{\infty} \frac{a_n (n+1)_p 2^r (n)_r j^{n+p+r}}{p!\, r!} L_{n+p}^{(\alpha-p)}(x)\, C_s^{n+r}(u) z^n \quad \ldots\ldots\ldots (6.3.4)$$

This gives the following:

Corollary

If there exists a bilateral generating function

$$G(x,u,j)=\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)\, C_s^n(u)\, j^n \qquad \ldots\ldots\ldots (6.3.5)$$

then

$$(1+j)^{\alpha}\exp(-jx)\ (1-2j)^{-s/2}\ G\left[x(1+j),\frac{u}{\sqrt{1-2j}},\frac{jz}{1-2j}\right]$$

$$=\sum_{n,p,r=0}^{\infty}\frac{a_n\,(n+1)_p\,2^r\,(n)_r\,j^{n+p+r}}{p!\,r!}\,L_{n+p}^{(\alpha-p)}(x)\,C_s^{n+r}(u)\,z^n\ . \qquad \ldots\ldots\ldots (6.3.6)$$

(III) If we put $m = s = 0$ in Theorem (2), we obtain the following result

$$(1+j)^{\alpha}\exp(-jx-j)\ G\left[x(1+j),\frac{jz}{1-2j}\right]$$

$$=\sum_{n,p,q,=0}^{\infty} a_n\,\frac{j^{n+p+q}\,(n+1)_p\,(-1)^q}{p!\,q!}\,(1-2j)^{-n}\,L_{n+p}^{(\alpha-p)}(x)\,z^n \qquad \ldots\ldots\ldots (6.3.7)$$

Replacing $\frac{z}{1-2j}$ by z, we obtain

$$(1+j)^{\alpha}\exp(-j)\ \exp(-jx)G\left[x(1+j),jz\right]$$

$$=\exp(-j)\sum_{n,p=0}^{\infty} a_n\frac{j^{n+p}\,(n+1)_p}{p!}\,L_{n+p}^{(\alpha-p)}(x)\,z^n$$

Thus, we have

$$(1+j)^{\alpha}\ \exp(-jx)G\left[x(1+j),jz\right]$$

$$= \sum_{n,p=0}^{\infty} a_n \frac{j^{n+p}(n+1)_p}{p!} L_{n+p}^{(\alpha-p)}(x) z^n$$

$$= \sum_{n,=0}^{\infty} \sum_{,p=0}^{n} a_{n-p} \frac{j^n (n-p+1)_p}{p!} L_n^{(\alpha-p)}(x) z^{n-p}$$

$$= \sum_{n,=0}^{\infty} j^n \sum_{,p=0}^{n} a_{n-p} \binom{n}{p} L_n^{(\alpha-p)}(x) z^{n-p}$$

$$= \sum_{n=0}^{\infty} j^n \sigma_n (x,z) \qquad \ldots\ldots\ldots (6.3.8)$$

where

$$\sigma_n (x,z) = \sum_{m=0}^{n} a_m \binom{n}{m} L_n^{(\alpha-n+m)}(x) z^m$$

which is the Theorem (1) {cf. [45], p.195}.

(IV) In our Theorem (2) if we put $m = 0$ and use

$$\sum_{r=0}^{\infty} \frac{(-w)^r}{r!} = e^{-w},$$

we notice that $G\ (x,\ u,\ z,\ w)$ becomes $G\ (x,\ u,\ w)$ for $L_0^{(n)}(z) = 1$.

Hence our Theorem (2) becomes

If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n w^n P_n(x) C_s^n(u) \qquad \ldots\ldots\ldots (6.3.9)$$

then

$$(1+2wx+w^2)^{-1/2}\ (1-2w)^{-s/2}$$

$$G\left(\frac{x+w}{\sqrt{1+2wx+w^2}}, \frac{u}{\sqrt{1-2w}}, \frac{wj}{(1-2w)\sqrt{1+2wx+w^2}}\right)$$

$$= \sum_{n,p,q=0}^{\infty} \frac{a_n\,(-1)^p\,(n+1)_p\,2^q\,(n)_q\,w^{n+p+q}}{p!\,q!} P_{n+p}(x)\,C_s^{n+q}(u)\,j^n \quad \ldots\ldots\ (6.3.10)$$

which was derived by Kar and Basu {[32], p.448}.

(V) In Theorem (2) we set s = 0, and use

$$\sum_{q=0}^{\infty}\frac{(n)_q}{q!}(2w)^q = (1-2w)^{-n},$$

we obtain

$$(1+2wx+w^2)^{-1/2}\ e^{-w}\ G\left[\frac{x+w}{\sqrt{1+2wx+w^2}}, z+w, \frac{wj}{(1-2w)\sqrt{1+2wx+w^2}}\right]$$

$$= \sum_{n,p,r=0}^{\infty} \frac{a_n\,(-1)^{p+r}\,(n+1)_p\,w^{n+p+r}}{p!\,r!} P_{n+p}(x)\,L_m^{n+r}(z)\ j^n\,(1-2w)^{-n}$$

$$\ldots\ldots\ (6.3.11)$$

Replacing $\frac{j}{1-2w}$ by j, we obtain

$$(1+2wx+w^2)^{-1/2}\ e^{-w}\ G\left[\frac{x+w}{\sqrt{1+2wx+w^2}}, z+w, \frac{wj}{\sqrt{1+2wx+w^2}}\right]$$

$$= \sum_{n,p,r}^{\infty} \frac{a_n\,(-1)^{p+r}\,(n+1)_p\,w^{n+p+r}}{p!\,r!} P_{n+p}(x)\,L_m^{n+r}(z)\,j^n. \quad \ldots\ldots\ (6.3.12)$$

This gives the following:

Corollary

If there exists a bilateral generating function

$$G(x,u,z,w)=\sum_{n=0}^{\infty} a_n w^n\, P_n(x)\; L_m^{(n)}(z) \qquad \ldots\ldots\ldots (6.3.13)$$

then

$$(1+2wx+w^2)^{-1/2}\; e^{-w}\; G\left[\frac{x+w}{\sqrt{1+2wx+w^2}},\, z+w,\, \frac{wj}{\sqrt{1+2wx+w^2}}\right]$$

$$=\sum_{n,p,r}^{\infty} \frac{a_n\,(-1)^{p+r}\,(n+1)_p\; w^{n+p+r}}{p!\;r!}\; P_{n+p}(x)\; L_m^{n+r}(z)\, j^n. \qquad \ldots\ldots\ldots (6.3.14)$$

(VI) If we put m = s = 0, our Theorem (2) becomes

$$\text{If } G(x,w)=\sum_{n=0}^{\infty} a_n\, w^n\, P_n(x) \qquad \ldots\ldots\ldots (6.3.15)$$

then

$$(1+2wx+w^2)^{-1/2}\; e^{-w}\; G\left[\frac{x+w}{\sqrt{1+2wx+w^2}},\, \frac{wj}{(1-2w)\sqrt{1+2wx+w^2}}\right]$$

$$= e^{-w} \sum_{n,p=0}^{\infty} \frac{a_n\,(-1)^p\,(n+1)_p\; w^{n+p}}{p!}\; P_{n+p}(x)\;\; j^n\,(1-2w)^{-n} \qquad \ldots\ldots\ldots (6.3.16)$$

Replacing $\dfrac{j}{1-2w}$ *by j*, we obtain

$$(1+2wx+w^2)^{-1/2}\; G\left[\frac{x+w}{\sqrt{1+2wx+w^2}},\, \frac{wj}{\sqrt{1+2wx+w^2}}\right]$$

$$= \sum_{n,p=0}^{\infty} \frac{a_n (-1)^p (n+1)_p w^{n+p}}{p!} P_{n+p}(x) j^n$$

$$= \sum_{p=0}^{\infty} \sum_{n=0}^{p} \frac{a_n (-1)^{p-n} (n+1)_{p-n} w^p}{(p-n)!} P_p(x) j^n \qquad \text{........ (6.3.17)}$$

which is the Theorem (1) {cf.[32], p.447}.

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